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Around Polygons in $\mathbb{R}^3$ and S^3 ⋆

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Abstract: We survey certain moduli spaces in low dimensions and some of the geometric structures that they carry, and then construct identifications among all of these spaces. In particular, we identify the moduli spaces of polygons in $\mathbb{R}^3$ and S^3, the moduli space of restricted representations of the fundamental group of a punctured 2-sphere, the moduli space of flat connections on a punctured sphere, the moduli space of parabolic bundles on a sphere, the moduli space of weighted points on $\mathbb{CP}^1$ and the symplectic quotient of $SO(3)$ acting diagonally on $(S^2)^n$. All of these spaces depend on parameters and some the above identifications require the parameters to be small. One consequence of this work is that these spaces are all biholomorphic with respect to the most natural complex structures they can each be given.

1. Introduction

In this paper we shall describe a coincidence that occurs among a number of moduli spaces of geometric objects in two, three, and infinite dimensions. These spaces arise in a series of very simple but apparently quite unrelated problems, and themselves carry a variety of geometric structures. Despite their disparate origins, we shall exhibit explicit maps identifying all of the spaces and hence show that they each share all of the geometric structures of their siblings.

Let us here at least name the main spaces which we shall go on to identify and give a diagram displaying the maps we shall construct among them. For a vector $\boldsymbol{s} = (s_1, \dots, s_n)$ of positive real numbers with $\sum_{j=1}^n s_j < 1$, we shall consider the following spaces, whose precise definitions and technical details shall be discussed in the following sections:

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- the space $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$ of configurations of a polygon with side lengths $\boldsymbol{s}$ in Euclidean 3-space; and the corresponding moduli space $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3)$ under the action of the group of orientation-preserving Euclidean motions;
- the symplectic manifold $(S^2)^n_{\boldsymbol{s}}$ consisting of the product of n spheres, where the j^{th} has the usual symplectic form scaled by s_j; and its reduction $(S^2)^n_{\boldsymbol{s}} /\!\!/ SO(3)$ by the action of $SO(3)$;
- the configuration space $\mathcal{W}_{\boldsymbol{s}}^{ss}$ of n points on $\mathbb{CP}^1$, semi-stable with respect to the weights $\boldsymbol{s}$; and the corresponding geometric invariant theory quotient $\mathcal{W}_{\boldsymbol{s}}^{ss}/\sim$;
- $\mathcal{V}_{\boldsymbol{s}}^{ss}$, the space of semi-stable rank two parabolic vector bundles on $\mathbb{CP}^1$ of degree zero, parabolic degree zero and having parabolic weights $-s_j$ and s_j at the j^{th} parabolic point; and the moduli space $\mathcal{V}_{\boldsymbol{s}}^{ss}/\sim$;
- the space $\mathcal{C}_{\boldsymbol{s},\delta}^{ss}$ of semi-stable, $L^2_{1,\delta}$ complex structures in a trivial rank two Hermitian vector bundle over the n-punctured sphere which near the j^{th} puncture are asymptotic to $\bar\partial + \frac{d\bar z}{2\bar z} \otimes \begin{pmatrix} -s_j/2 & 0 \\ 0 & s_j/2 \end{pmatrix}$; and the moduli space $\mathcal{C}_{\boldsymbol{s},\delta}^{ss}/\sim$;
- the set $\mathcal{F}_{\boldsymbol{s},\delta}$ of flat unitary connections in $\mathcal{C}_{\boldsymbol{s},\delta}$; and the quotient $\mathcal{F}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\delta}$ by the group of special unitary $L^2_{2,\delta}$ gauge transformations asymptotic to elements of the maximal torus $T \subset SU(2)$ at each puncture;
- the set $\mathcal{R}_{\boldsymbol{s}}$ of those representations of the fundamental group of the n-punctured sphere into $SU(2)$ which take the loop around the j^{th} puncture to elements of $SU(2)$ with trace $2\cos(\pi s_j)$; and the corresponding moduli space $\mathcal{R}_{\boldsymbol{s}}/SU(2)$ under the conjugation action of $SU(2)$;
- the space $\mathcal{P}_{\pi\boldsymbol{s}}^{S^3}$ of configurations of a geodesic polygon with side lengths $\pi\boldsymbol{s}$ in the sphere S^3; and the corresponding moduli space $\mathcal{P}_{\pi\boldsymbol{s}}^{S^3}/SO(4)$ under the action of the isometry group of the sphere.

Since there are so many configuration and moduli spaces appearing in this paper, we have attempted to use a somewhat suggestive notation: each space has as representative symbol the first letter of a word that describes the objects in question. Hence we use $\mathcal{P}$ for polygons, $\mathcal{W}$ for weighted points, $\mathcal{V}$ for vector bundles, $\mathcal{C}$ for both complex structures and connections, $\mathcal{F}$ for flat connections and $\mathcal{R}$ for representations.

With this notation understood, it now makes sense to say that our main purpose here is to fill in the arrows in the following diagram:

$$\begin{tikzcd}
\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3) \arrow[r] & (S^2)^n_{\boldsymbol{s}} /\!\!/ SO(3) \arrow[r] & \mathcal{W}_{\boldsymbol{s}}^{ss}/\sim \arrow[r] & \mathcal{V}_{\boldsymbol{s}}^{ss}/\sim \arrow[d, equal] \\
\mathcal{P}_{\pi\boldsymbol{s}}^{S^3}/SO(4) \arrow[r] & \mathcal{R}_{\boldsymbol{s}}/SU(2) & \mathcal{F}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\delta} \arrow[l] & \mathcal{C}_{\boldsymbol{s},\delta}^{ss}/\sim \arrow[l]
\end{tikzcd}$$

In fact, the remaining sections of this paper are nothing other than a grand tour of the above diagram; sections alternate, where appropriate, between describing spaces and some of their properties and constructing maps between these spaces.

Note that in the above diagram, $\mathcal{W}_{\boldsymbol{s}}^{ss}/\sim$ and $\mathcal{V}_{\boldsymbol{s}}^{ss}/\sim$ are naturally complex spaces and, as we shall show below, the map connecting them is a biholomorphism with respect to these structures. It shall thus follow that the above diagram consists of complex isomorphisms when all of the spaces are endowed with any of the available holomorphic structures.

We should mention that this is by no means the first paper to discuss these spaces, their structures or the existence of maps between some of them; our purpose is in fact to bring together this material in one place and to fill in a number of the missing connections between spaces and structures. In particular, Sects. 3–6 describe results of Kapovich and Millson [12], Sects. 7, 8 and part of Sect. 9 are entirely new to this paper, the rest of Sect. 9 and Sect. 10 are closely related to the work of Poritz in [16], while Sects. 11 and 12 are based on a special case of Kapovich and Millson's [11]; work of other authors on these topics is also mentioned in the references of [12], [11] and [16]. A concluding Sect. 13 summarizes what we have done, in a more accurate and complete diagram than the above, and mentions two further, more recent, related works.

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2. Configuration Spaces of Polygons in $\mathbb{R}^3$: $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3)$

Let $n \geq 3$ be an integer and $\boldsymbol{s} = (s_1, \ldots, s_n)$ an n-tuple of positive real numbers. Then

Definition 2.1. *The* **configuration space** $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$ **of polygons in** $\mathbb{R}^3$ **with fixed side lengths** $\boldsymbol{s}$ *is the set of all n-tuples $(e_1, \ldots, e_n)$ of directed line segments such that the length of e_j is s_j and the endpoint of e_j is the beginning point of $e_{j+1 (\mathrm{mod}\ n)}$. The* **moduli space of polygons** *is then simply $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3)$, where $\mathbb{E}^+(3)$ acts diagonally on the n-tuples of line segments.*

Note that it might actually be more precise to call these "labeled polygons", since we are keeping track of which line segment is first, which is second, *etc*. Hence two polygons will be identified by an element $g \in \mathbb{E}^+(3)$ if and only if g moves one polygon exactly onto the other, maintaining the numbering of sides. Of course, for generic choices of side lengths $\boldsymbol{s}$, the only identification possible between elements of $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$ would have to match corresponding sides.

Let us remark that the vector $\boldsymbol{s}$ of side-lengths of our polygons can be scaled by any positive real number and it will have an obvious effect on the configuration and moduli spaces. In fact, it shall be necessary in Sect. 8, below, to assume that $\sum_{j=1}^{n} s_j < 1$, but for the time being we need make no such restriction.

There are a number of equivalent ways to give $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$ the structure of a smooth manifold. One very direct method goes as follows: each segment e_j is defined by a pair of points $(e_j^{\mathrm{begin}}, e_j^{\mathrm{end}}) \in S_j$, where $S_j = \{(p, q) \mid d(p, q) = s_j\} \subset \mathbb{R}^3 \times \mathbb{R}^3$. Each of these S_j's is a smooth manifold – in fact, a trivial S^2 bundle over either $\mathbb{R}^3$ factor – and $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$ can be identified with the set $f^{-1}((0, \ldots, 0))$, where $f : S_1 \times \cdots \times S_n \to \left(\mathbb{R}^3\right)^n$: $((p_1, q_1), \ldots, (p_n, q_n)) \mapsto (p_2 - q_1, \ldots, p_1 - q_n)$. The zero set of f will be smooth if we avoid certain polygons:

Definition 2.2. *A polygon is said to be* **degenerate** *if it lies entirely in some line in $\mathbb{R}^3$. The set of non-degenerate polygons shall be denoted $\widehat{\mathcal{P}}_{\boldsymbol{s}}^{\mathbb{R}^3}$.*

Now an application of the inverse function theorem with f shows that $\widehat{\mathcal{P}}_{\boldsymbol{s}}^{\mathbb{R}^3}$ is a smooth submanifold of $S_1 \times \cdots \times S_n$.

An additional feature of the space of non-degenerate polygons is that $\mathbb{E}^+(3)$ acts freely there, while the degenerate polygons all have stabilizer isomorphic to $SO(2)$. This means that $\widehat{\mathcal{P}}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3)$ is a smooth manifold, while on all of $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}/\mathbb{E}^+(3)$ we can at least use the quotient topology. Note, however, that for generic side lengths $\boldsymbol{s}$, there are no degenerate polygons whatsoever in $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$, while even if there are some in a $\mathcal{P}_{\boldsymbol{s}}^{\mathbb{R}^3}$, they form a closed, $\mathbb{E}^+(3)$-invariant set of high codimension.

When a compact group acts freely on a manifold it is particularly easy to give the quotient space a manifold structure. This motivates passing to

Definition 2.3. *The set of* **based polygons** *is* $\mathcal{P}_{s,0}^{\mathbb{R}^3} = \{(e_1, \ldots, e_n) \in \mathcal{P}_s^{\mathbb{R}^3} \mid e_1^{\text{begin}} = 0 \in \mathbb{R}^3\}$.

Then $\mathcal{P}_s^{\mathbb{R}^3}/\mathbb{E}^+(3)$ and $\mathcal{P}_{s,0}^{\mathbb{R}^3}/SO(3)$ are homeomorphic, as are $\widehat{\mathcal{P}}_s^{\mathbb{R}^3}/\mathbb{E}^+(3)$ and $\widehat{\mathcal{P}}_{s,0}^{\mathbb{R}^3}/SO(3)$, and this last space is a smooth manifold.

3. The First Gauss Map

Let us begin transforming the configuration space of polygons in $\mathbb{R}^3$ and its moduli space. The essential idea here is to build a sort of discrete Gauss map – that is, to assign to each edge of a polygon the unit vector in that direction. So consider the map

$$\overline{G}_1 : \mathcal{P}_{s,0}^{\mathbb{R}^3} \to (S^2)^n : (\ldots, e_j, \ldots) \mapsto (\ldots, \frac{e_j^{\text{end}} - e_j^{\text{begin}}}{s_j}, \ldots).$$

To invert $\overline{G}_1$, we start with some $(u_1, \ldots, u_n) \in (S^2)^n$ and build edges by setting $e_1^{\text{begin}} = 0$ and then inductively $e_j^{\text{end}} = e_j^{\text{begin}} + s_j u_j$ and $e_{j+1}^{\text{begin}} = e_j^{\text{end}}$. This will give us a well-defined closed polygon in $\mathcal{P}_{s,0}^{\mathbb{R}^3}$ as long as the end of the last edge is the beginning of the first, i.e., if

$$\begin{aligned} e_1^{\text{begin}} &= 0 = e_n^{\text{end}} = e_n^{\text{begin}} + s_n u_n = e_{n-1}^{\text{end}} + s_n u_n \\ &= \cdots = e_1^{\text{begin}} + s_1 u_1 + \cdots + s_n u_n = s_1 u_1 + \cdots + s_n u_n. \end{aligned}$$

Hence if we define $\mu_s : (S^2)^n \to \mathbb{R}^3 : (u_1, \ldots, u_n) \mapsto s_1 u_1 + \cdots + s_n u_n$, then $\overline{G}_1$ is an $SO(3)$-equivariant homeomorphism of $\mathcal{P}_{s,0}^{\mathbb{R}^3}$ with $\mu_s^{-1}(0)$ and consequently also induces a diffeomorphism G_1 of $\widehat{\mathcal{P}}_{s,0}^{\mathbb{R}^3}/SO(3)$ with $\widehat{\mu_s^{-1}(0)}/SO(3)$, where $\widehat{\mu_s^{-1}(0)} = \overline{G}_1(\widehat{\mathcal{P}}_{s,0}^{\mathbb{R}^3})$.

4. Symplectic Quotients of Products of Spheres: $(S^2)^n_s /\!\!/ SO(3)$

Our choice of the letter μ_s for the map defined above was not an accident: it is nothing other than the moment map for the diagonal action of $SO(3)$ on $(S^2)^n$, where the latter is given the symplectic structure $s_1\pi_1^*(\text{vol}) + \cdots + \mathrm{s_n}\pi_{\mathrm{n}}^*(\text{vol})$. Here the π_j are the various projections onto the factors, vol is the standard volume form on the sphere (of total volume 4π) and the target $\mathbb{R}^3$ is to be thought of as the dual, via the usual inner product, of $(\mathbb{R}^3, \times) \cong (\mathfrak{so}(3), [\ ,\ ])$, where $\times$ is the cross-product.

Definition 4.1. *We write* $(S^2)^n_s$ *for* $(S^2)^n$ *with the above symplectic structure, and set* $\widehat{(S^2)^n_s} = \{(u_1, \ldots, u_n) \in (S^2)^n_s \mid \text{not all } u_i = \pm u_j\}$.

It now follows that the map G_1 of the last section is in fact a homeomorphism of $\mathcal{P}_{s,0}^{\mathbb{R}^3}/SO(3)$ with the Marsden–Weinstein symplectic reduction $(S^2)^n_s /\!\!/ SO(3) = \mu_s^{-1}(0)/SO(3)$. Similarly, $\widehat{\mu_s^{-1}(0)} = \mu_s^{-1}(0) \cap \widehat{(S^2)^n_s}$ and G_1 is a diffeomorphism of the smooth manifold $\widehat{\mathcal{P}}^{\mathbb{R}^3}_{\boldsymbol{s},0}/SO(3)$ with the symplectic manifold $\widehat{(S^2)^n_{\boldsymbol{s}}}/\!\!/SO(3)$. Furthermore, as the smooth symplectic reduction of a Kähler manifold, $\widehat{(S^2)^n_{\boldsymbol{s}}}/\!\!/SO(3)$ is itself Kähler, [7]. Kapovich and Millson have in fact shown that $(S^2)^n_{\boldsymbol{s}}/\!\!/SO(3)$ can be given a $\mathbb{C}$-analytic structure even near its singular stratum, see [12] for details.

5. The First Kempf–Ness-Type Theorem

We can also think of the symplectic manifold S^2 as the complex algebraic variety $\mathbb{CP}^1$, and the action of $SO(3)$ on S^2 extends to the algebraic action of the group $PSL(2,\mathbb{C})$ of biholomorphisms of $\mathbb{CP}^1$. Certainly then the inclusions $SO(3) \hookrightarrow PSL(2,\mathbb{C})$ and $\mu_{\boldsymbol{s}}^{-1}(0) \hookrightarrow (\mathbb{CP}^1)^n$ induce a map

$$\kappa_1 : (S^2)^n_{\boldsymbol{s}}/\!\!/SO(3) \to (\mathbb{CP}^1)^n/PSL(2,\mathbb{C}).$$

In situations such as the current one, it often turns out that this map is a bijection onto the quotient of a non-empty Zariski open in the target; this is the original theorem of Kempf and Ness [13], much elaborated by Kirwan, [14]. In fact, Kirwan's result applies to our spaces, but we shall instead give the proof of [12] which can be interpreted in our present context much more directly and concretely.

For this it is instructive to think of the points of $(S^2)^n_{\boldsymbol{s}}$ as giving purely atomic measures on the sphere, where $(u_1,\ldots,u_n) \in (S^2)^n_{\boldsymbol{s}}$ corresponds to the measure $\sum_{j=1}^n s_j\delta_{u_j}$. We have the standard

Definition 5.1. *For any measure ν on S^2, the* **center of mass of** ν *is*

$$B(\nu) = \int_{S^2} x\,d\nu(x),$$

where $x \in S^2 \hookrightarrow \mathbb{R}^3$ and the integral is of this vector-valued function.

Note $B(\nu)$ is always in the closed unit ball $\overline{B(0,\nu(S^2))} \subset \mathbb{R}^3$ and only on $\partial B(0,\nu(S^2))$ if ν is concentrated at a point. There is another center of mass $C(\nu)$, whose definition is in the work [6] of Douady and Earle, which is called the *conformal center of mass* and satisfies:

- $C(g_*\nu) = g(C(\nu))$ for any $g \in PSL(2,\mathbb{C})$, and
- $C(\nu) = 0$ if and only if $B(\nu) = 0$,

but which is only defined for stable measures ν, where

Definition 5.2. *A measure ν on S^2 is said to be* **stable** *(respectively,* **semi-stable***) if the mass of any atom is less than (respectively, less than or equal to) $\frac{1}{2}\nu(S^2)$.*

This is exactly the tool we need. The measures corresponding to $\widehat{\mu_{\boldsymbol{s}}^{-1}(0)}$ are stable and have center of mass at 0. If an element $g \in PSL(2,\mathbb{C})$ leaves the center of mass of a stable measure at 0, then it also fixes the conformal center of mass at 0 and hence must lie in the stabilizer $SO(3)$ of 0 in $PSL(2,\mathbb{C})$. Conversely, by the transitivity of $PSL(2,\mathbb{C})$ on $B(0,1)$, any stable purely atomic measure can be moved until its conformal center of mass, hence also its normal center of mass, is at 0. Thus κ_1 is a bijection of $\widehat{(S^2)^n_{\boldsymbol{s}}}/\!\!/SO(3)$ with the part of $(\mathbb{CP}^1)^n/PSL(2,\mathbb{C})$ corresponding to orbits of stable measures.

6. Weighted Quotients of Points on $\mathbb{CP}^1$: $\mathcal{W}_s^{ss}/\sim$

It turns out that this image of κ_1 is exactly the stable part of the weighted quotient by $PSL(2,\mathbb{C})$ of the configuration space of n points on $\mathbb{CP}^1$ studied by Deligne and Mostow in [4]. One can motivate this construction as follows: Let us for a moment write $\mathcal{W}_0 \subset (\mathbb{CP}^1)^n$ for the set of n-tuples of distinct points in $\mathbb{CP}^1$. Then $\mathcal{W}_0$ is a quasi-projective $\mathbb{C}$-algebraic variety and is closed under the free, diagonal action of $PSL(2,\mathbb{C})$; the quotient $\mathcal{W}_0/PSL(2,\mathbb{C})$ is a smooth quasi-projective variety. The complication comes when we try to complete $\mathcal{W}_0$ by putting back in some of the $PSL(2,\mathbb{C})$-orbits in $(\mathbb{CP}^1)^n \smallsetminus \mathcal{W}_0$ before taking the quotient.

Deligne and Mostow [4] give a geometric invariant theory approach to this problem, depending on a choice of a vector $\boldsymbol{s}$ of positive real numbers. Using the terminology we have developed above, we can restate their definitions as follows:

Definition 6.1. *Let the subsets $\mathcal{W}_s^s$ and $\mathcal{W}_s^{ss}$ of* **stable** *and* **semi-stable weighted points in** $(\mathbb{CP}^1)^n$ *be those points $(u_1,\ldots,u_n) \in \mathcal{W} = (\mathbb{CP}^1)^n$ such that the corresponding measure $\sum_{j=1}^n s_j\delta_{u_j}$ is stable and semi-stable, respectively; also write $\mathcal{W}_s^{cusp} = \mathcal{W}_s^{ss} \smallsetminus \mathcal{W}_s^s$.*

There is the usual geometric invariant theory equivalence relation $\sim$ on $\mathcal{W}_s^{ss}$ which on $\mathcal{W}_s^s$ is the relation given by the $PSL(2,\mathbb{C})$-orbits and on $\mathcal{W}_s^{\mathrm{cusp}}$ is *extended orbit equivalence*, i.e., two points are equivalent if and only if their orbit closures intersect. Then $\mathcal{W}_s^s/\sim\, = \mathcal{W}_s^s/PSL(2,\mathbb{C})$ can be given the structure of a smooth, quasi-projective $\mathbb{C}$-algebraic variety and $\mathcal{W}_s^{ss}/\sim$ that of a projective variety; if all of the s_j are rational, these are simply the geometric invariant theory quotients à la Mumford.

One pleasant feature of the geometric invariant theory in this application is the existence of some particularly useful semi-stable points:

Definition 6.2. *The* **nice semi-stable** *points $\mathcal{W}_s^{nss} \subset \mathcal{W}_s^{ss}$ are those whose $PSL(2,\mathbb{C})$-orbit is closed in $\mathcal{W}_s^{ss}$.*

Considering the action of $PSL(2,\mathbb{C})$ on $\mathbb{CP}^1$, we see that the points of $\mathcal{W}_s^{nss} \cap \mathcal{W}_s^{\mathrm{cusp}}$ correspond to measures with exactly two atoms, each having half the total mass. What makes these so nice is the fact that inclusion induces a bijection $\mathcal{W}_s^{ss}/\sim\, = \mathcal{W}_s^{nss}/PSL(2,\mathbb{C})$. Under κ_1^{-1}, $(\mathcal{W}_s^{nss} \cap \mathcal{W}_s^{\mathrm{cusp}})/PSL(2,\mathbb{C})$ corresponds exactly to $(S^2)_s^n /\!\!/ SO(3) \smallsetminus \widehat{(S^2)_s^n} /\!\!/ SO(3)$, i.e., to G_1 of the degenerate polygons. Thus κ_1 is a homeomorphism of $(S^2)_s^n /\!\!/ SO(3)$ with the projective $\mathbb{C}$-algebraic variety $\mathcal{W}_s^{ss}/\sim$ which identifies the corresponding smooth and singular parts. It is in fact easy to check that $\kappa_1\big|_{\widehat{(S^2)_s^n}/SO(3)}$ is smooth, while in [12] it is also shown that their $\mathbb{C}$-analytic structure near the singular set is mapped analytically by κ_1 to the complex structure near $\mathcal{W}_s^{\mathrm{cusp}}/\sim$.

7. The Passage to Vector Bundles

Fix now a trivial holomorphic vector bundle E of rank two over $\mathbb{CP}^1$ (not the same $\mathbb{CP}^1$ as in previous sections) and $n+1$ points $p_1,\ldots,p_n,q \in \mathbb{CP}^1$. Since E is trivial there is a canonical identification of each of the fibers $E\big|_{p_j}$ with the fiber $E\big|_q$ over q and choosing an isomorphism $\mathbb{C}^2 \cong E\big|_q$, we will thus have an identification of the points of $\mathbb{CP}^1$ (the $\mathbb{CP}^1$ of previous sections, this time) with the set of lines in $E\big|_q$, and hence with the set of lines in each $E\big|_{p_j}$. We shall write $v \leftrightsquigarrow \mathcal{L}$ if this identification matches a $v \in \mathbb{CP}^1$ with the line $\mathcal{L} \subset E\big|_{p_j}$.

With the above choices and identifications in place and still using our n-vector $\boldsymbol{s}$ of real numbers, we can define for every $(v_1, \dots, v_n) \in (\mathbb{CP}^1)^n$ some additional structure on E as follows. For each p_j, we define a filtration $E\big|_{p_j} = E_j^1 \supsetneq E_j^2 \supsetneq 0$, where $v_j \leftrightsquigarrow E_j^2$ is the only interesting filtration step; we also attach to E_j^1 the number $-s_j$ and to E_j^2 the number s_j and call these the *weights* of these steps of the filtration.

Suppose we now consider a holomorphic automorphism g of E of trivial determinant. Since E is trivial over a compact base, g is constant with respect to any holomorphic trivialization. Letting $h \in SL(2, \mathbb{C})$ be the matrix of $g\big|_q$ with respect to our identification $\mathbb{C}^2 \cong E\big|_q$, we see that the action of g on E takes the filtrations E_j^k coming from $(v_1, \dots, v_n) \in (\mathbb{CP}^1)^n$ to those coming from $(hv_1, \dots, hv_n)$. Thus our map from $(\mathbb{CP}^1)^n$ to trivial vector bundles with filtration and weight data sends orbits under $PSL(2, \mathbb{C})$ to orbits under the group of holomorphic automorphisms of E which are trivial on $\det E$.

Let us check what sort of data arises on E when we start with a point in $\mathcal{W}_{\boldsymbol{s}}^s$ or $\mathcal{W}_{\boldsymbol{s}}^{ss}$. So note that the mass of an atom at some point $v \in \mathbb{CP}^1$ of the measure corresponding to a $(v_1, \dots, v_n) \in (\mathbb{CP}^1)^n$ is

$$\sum_{j \,|\, v = v_j} s_j = \sum_{j \,|\, v \leftrightsquigarrow E_j^2} s_j = \sum_{j \,|\, L^v|_{p_j} = E_j^2} s_j, \tag{7.1}$$

where L^v is the unique line subbundle of E with $L^v\big|_q = v$ that is constant with respect to a holomorphic trivialization of E, i.e., the line subbundle of degree zero whose fiber at q is v. It is then appropriate to make the

Definition 7.1. *Given a bundle E with filtration and weight data as above we set for any line subbundle L of E the* **mass of** *L to be the number*

$$\sum_{j \,|\, L|_{p_j} = E_j^2} s_j.$$

We can now say that the filtration and weight data on E that results from a point of $\mathcal{W}_{\boldsymbol{s}}^s$ (respectively, $\mathcal{W}_{\boldsymbol{s}}^{ss}$) is that data such that for all holomorphic line subbundles $L \subset E$ of degree zero the mass of L is less than (respectively, less than or equal to) $\frac{1}{2}\sum_{j=1}^n s_j$.

8. Moduli of Parabolic Vector Bundles: $\mathcal{V}_{\boldsymbol{s}}^{ss}/\!\sim$

The decorated vector bundles that appeared in the last section have been studied before.

Definition 8.1. *Let F be a holomorphic vector bundle over a compact Riemann surface Σ with n distinguished points $p_1, \dots, p_n \in \Sigma$. A* **parabolic structure on** *F is the choice for each j of*

- *a decreasing flag $F\big|_{p_j} = F_j^1 \supsetneq \cdots \supsetneq F_j^{k_j} \supsetneq F_j^{k_j+1} = 0$ in the fiber over p_j; and*
- *an increasing sequence $\alpha_1^j < \cdots < \alpha_{k_j}^j$ of real numbers, called* **weights**, *one for each step of the filtration at p_j.*

The weight α_ℓ^j *is said to occur with* **multiplicity** $m_\ell^j = \dim F_j^\ell / F_j^{\ell+1}$ *and the* **parabolic degree** *and* **parabolic slope** *of* (F, α) *are*

$$\text{par-deg}(F, \alpha) = \deg F + \sum_{j=1}^{n} \sum_{\ell=1}^{k_j} m_\ell^j \alpha_\ell^j$$

and

$$\mu_{\text{par}}(F, \alpha) = \frac{\text{par-deg}(F, \alpha)}{\text{rank}\, F}.$$

An **isomorphism of parabolic vector bundles** *is an isomorphism of holomorphic bundles which takes one parabolic structure exactly to another.*

This definition was first given by Mehta and Seshadri in [15], but with the additional requirement that all of the weights lie in $[0, 1)$; the version here follows the constructions in [16], but it is easy to see how to generalize essentially all known results on parabolic vector bundles, *mutatis mutandis,* to this slightly larger context.

Given a holomorphic subbundle G of a parabolic vector bundle (F, α), there is a natural way to induce a parabolic structure on G. For each point p_j, we let the filtration of $G\,\big|_{p_j}$ be the intersection of $G\,\big|_{p_j}$ with the filtration of $F\,\big|_{p_j}$, with repetitions removed, and at the ℓ^{th} filtration step we assign the weight $\beta_\ell^j = \alpha_{\ell'}^j$, where ℓ' is the largest index such that $G_j^\ell \subset F_j^{\ell'}$. By convention, when we speak of a parabolic subbundle, it shall be assumed to have arisen in this way.

There is a good geometric invariant theory moduli space for vector bundles with parabolic structure, and it relies on the following

Definition 8.2. *A parabolic vector bundle* (F, α) *is* **stable** *(respectively,* **semi-stable***) if* $\mu_{\text{par}}(G, \beta) < \mu_{\text{par}}(F, \alpha)$ *(respectively,* $\mu_{\text{par}}(G, \beta) \leq \mu_{\text{par}}(F, \alpha)$*) for all proper parabolic subbundles* (G, β) *of* (F, α).

We can now concentrate on the sets of vector bundles which are relevant to our present study. So fix n points $p_1, \ldots, p_n$ on $\Sigma = \mathbb{CP}^1$ and our usual vector $\boldsymbol{s}$ of positive real numbers.

Definition 8.3. *Denote by* $\mathcal{V}_{\boldsymbol{s}}$ *the set of parabolic vector bundles of rank two, trivial determinant and such that at each* p_j *the filtration has two steps with weights* $-s_j$ *and* s_j*. Write also* $\mathcal{V}_{\boldsymbol{s}}^s$ *and* $\mathcal{V}_{\boldsymbol{s}}^{ss}$ *for the stable and semi-stable parts of* $\mathcal{V}_{\boldsymbol{s}}$*, respectively, and* $\mathcal{V}_{\boldsymbol{s}}^{\text{cusp}}$ *for* $\mathcal{V}_{\boldsymbol{s}}^{ss} \smallsetminus \mathcal{V}_{\boldsymbol{s}}^s$*. We adopt the convention that isomorphisms of elements of* $\mathcal{V}_{\boldsymbol{s}}$ *must have trivial determinant and denote the resulting equivalence relation* "Iso^0".

Note that the trivial determinants and weights of opposite signs mean that the normal and parabolic degrees of bundles in $\mathcal{V}_{\boldsymbol{s}}$ are both zero. Also, our notation $\mathcal{V}_{\boldsymbol{s}}^{\text{cusp}}$ is non-standard; these are usually called *properly semi-stable* bundles.

Let us look at the parabolic degrees of line subbundles of elements of $\mathcal{V}_{\boldsymbol{s}}$. So say L is a proper holomorphic subbundle of the vector bundle F at some point of $\mathcal{V}_{\boldsymbol{s}}$. Then the induced filtration on $L\,\big|_{p_j}$ has only one step, but the weight assigned to that step depends on the relation of $L\big|_{p_j}$ to F_j^2: it is s_j if $L\big|_{p_j} = F_j^2$ and $-s_j$ otherwise. Hence

$$\begin{aligned} \text{par-deg}\, L &= \deg L + \left(\sum_{j \mid L|_{p_j} = F_j^2} s_j - \sum_{\text{the rest}} s_j \right) \\ &= \deg L + 2 \sum_{j \mid L|_{p_j} = F_j^2} s_j - \sum_{j=1}^{n} s_j. \end{aligned} \tag{8.1}$$

It follows that if $F \in \mathcal{V}_{\boldsymbol{s}}^{ss}$ and L is any proper holomorphic subbundle, then

$$\deg L = \text{par-deg}\, L + \sum_{j=1}^{n} s_j - 2 \sum_{j \mid L|_{p_j} = F_j^2} s_j \leq \sum_{j=1}^{n} s_j.$$

Hence if the vector $\boldsymbol{s}$ is chosen so that this last sum is less than 1 then the degree of L, being an integer, will have to be non-positive, and so F will in fact be semi-stable as a plain holomorphic bundle. Since all holomorphic bundles over $\mathbb{CP}^1$ of rank two and degree zero are of the form $\mathcal{O}(k) \oplus \mathcal{O}(-k)$, such a parabolic-semi-stable F must be holomorphically trivial. We shall thus assume for the remainder of this paper that

$$\sum_{j=1}^{n} s_j < 1. \tag{8.2}$$

The gist of the last section was to define maps which we may call

$$\overline{\xi} : (\mathbb{CP}^1)^n \to \mathcal{V}_{\boldsymbol{s}} \;\text{ and }\; \xi : (\mathbb{CP}^1)^n / PSL(2, \mathbb{C}) \to \mathcal{V}_{\boldsymbol{s}} / \operatorname{Iso}^0 .$$

If we restrict $\overline{\xi}$ to $\mathcal{W}_{\boldsymbol{s}}^{s}$ (respectively, to $\mathcal{W}_{\boldsymbol{s}}^{ss}$), then we get elements of $\mathcal{V}_{\boldsymbol{s}}$ whose underlying holomorphic bundle is trivial and has the property that every holomorphic line subbundle of degree zero has mass less than (respectively, less than or equal to) $\frac{1}{2} \sum_{j=1}^{n} s_j$. But by (8.1) this means that every such line subbundle has parabolic degree less than zero (respectively, less than or equal to zero). Also by (8.1) and (8.2), any holomorphic line subbundle of negative degree has negative parabolic degree. Since all line subbundles of the trivial bundle are of non-positive degree, it follows that $\overline{\xi}(\mathcal{W}_{\boldsymbol{s}}^{s}) \subset \mathcal{V}_{\boldsymbol{s}}^{s}$ and $\overline{\xi}(\mathcal{W}_{\boldsymbol{s}}^{ss}) \subset \mathcal{V}_{\boldsymbol{s}}^{ss}$.

Furthermore, it is clear from the role of the flag varieties in the constructions of [15, §4] that the points in $\mathcal{V}_{\boldsymbol{s}}^{s} / \operatorname{Iso}^0$ depend holomorphically upon the filtration data at $p_1, \ldots, p_n$ (we shall also reprove this fact much more directly in the next section). This means that ξ is in fact a biholomorphism of $\mathcal{W}_{\boldsymbol{s}}^{s} / PSL(2, \mathbb{C})$ with $\mathcal{V}_{\boldsymbol{s}}^{s} / \operatorname{Iso}^0$.

In order to extend ξ to be a map of the semi-stable moduli space, let us again take advantage of the nice semi-stable points. So given a point of $\mathcal{W}_{\boldsymbol{s}}^{nss}$ with corresponding parabolic vector bundle $F \in \mathcal{V}_{\boldsymbol{s}}$, note that the two atoms which each have half of the total mass correspond to distinct trivial line subbundles L_1 and L_2 which by (8.1) both have parabolic degree zero, and in fact $F = L_1 \oplus L_2$ as parabolic bundles. Recall the

Definition 8.4. *A parabolic vector bundle which is the direct sum of stable parabolic subbundles all of the same parabolic slope is called* **polystable**. *Write $\mathcal{V}_{\boldsymbol{s}}^{ps}$ for the set of polystable bundles in $\mathcal{V}_{\boldsymbol{s}}$.*

Of course $\mathcal{V}_s^s \subset \mathcal{V}_s^{ps} \subset \mathcal{V}_s^{ss}$, but the general semi-stable bundle is an extension of a line bundle of parabolic degree zero by another such, rather than merely the direct sum of two such line bundles.

However, the equivalence relation used in the algebro-geometric construction of the moduli space of semi-stable parabolic bundles is exactly the isomorphism of associated graded bundles (associated to the Harder-Narasimhan filtration of a holomorphic bundle by successive maximally destabilizing subbundles; see [1]). Hence every semi-stable bundle is equivalent to one which is polystable and we can extend ξ to a bijection

$$\begin{aligned}\mathcal{W}_s^{\text{cusp}}/\sim\, &\cong (\mathcal{W}_s^{nss} \cap \mathcal{W}_s^{\text{cusp}})/PSL(2,\mathbb{C})\\ &\xrightarrow{\xi} (\mathcal{V}_s^{ps} \cap \mathcal{V}_s^{\text{cusp}})/\operatorname{Iso}^0 \cong \mathcal{V}_s^{\text{cusp}}/\sim\end{aligned}$$

and a homeomorphism $\xi : \mathcal{W}_s^{ss}/\sim\, \to \mathcal{V}_s^{ss}/\sim$.

9. A Reinterpretation in Infinite Dimensions: $\mathcal{V}_s/\operatorname{Iso}^0 = \mathcal{C}_{s,\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$

In our presentation here, the issue of what exactly are the points of $\mathcal{V}_s$ or its quotients is somewhat elusive: let us now make these points more concrete. But rather than approaching the intricacies of the algebraic geometry of these spaces, we choose to represent them as infinite-dimensional affine spaces and the resulting quotients by infinite-dimensional Lie groups. This is quite standard, since the work of Atiyah and Bott [1] and before, so our main work here will be in incorporating the parabolic structures of our bundles.

First, since all holomorphic vector bundles of degree zero are isomorphic as smooth bundles, we shall work always in our trivial bundle $E = \mathbb{C}^2 \times \mathbb{CP}^1$.

Definition 9.1. *A* **complex structure on** *E is an operator $\bar\partial_A : \Omega^0(E) \to \Omega^{(0,1)}(E)$ satisfying the $\bar\partial$-Leibniz rule and inducing the standard $\bar\partial$ on the line bundle* $\det E$. *Equivalently, fixing the standard constant Hermitian structure on E, a* **(special) unitary connection** *is an operator $\mathrm{d}_A : \Omega^0(E) \to \Omega^1(E)$ satisfying the usual Leibniz rule and also preserving the metric and inducing the standard* d *on* $\det E$. *In either case, the space of such operators shall be denoted $\mathcal{C}$, and the subset of complex structures for which the resulting holomorphic bundle is stable shall be denoted $\mathcal{C}^s$.*

We get from one kind of operator to the other by the maps $\bar\partial_A = \bar\partial + A \mapsto \mathrm{d} + A - A^*$ and $\mathrm{d}_A = \mathrm{d} + A^{1,0} + A^{0,1} \mapsto \bar\partial + A^{1,0}$. While usually the space of holomorphic bundles is an affine space, our E is trivial and thus we have a natural basepoint $\bar\partial$. Therefore $\mathcal{C} = \{\bar\partial + \eta \mid \eta \in \mathfrak{sl}(2,\mathbb{C}) \otimes \Omega^{(0,1)}_{\mathbb{CP}^1}\}$ is essentially a complex vector space.

Next, we must consider when two complex structures are equivalent.

Definition 9.2. *Two operators $\bar\partial_A, \bar\partial_B \in \mathcal{C}$ are said to give isomorphic holomorphic bundles if and only if $\bar\partial_B = g^*(\bar\partial_A) = g^{-1} \circ \bar\partial_A \circ g$ for some g in the* **complex gauge group** *$\mathcal{G}^{\mathbb{C}}$ of smooth sections of the bundle* $\operatorname{SAut} E = SL(2,\mathbb{C}) \times \mathbb{CP}^1$ *of automorphisms of E with trivial determinant. The subgroup $\mathcal{G} \subset \mathcal{G}^{\mathbb{C}}$ of automorphisms which preserve the metric is called the* **(special) unitary gauge group**.

$\mathcal{C}^s$ is in fact an open, $\mathcal{G}^{\mathbb{C}}$-invariant subset of $\mathcal{C}$, so the moduli space of stable bundles is nothing other than $\mathcal{C}^s/\mathcal{G}^{\mathbb{C}}$, which inherits a complex structure as the quotient of a complex space acted upon holomorphically by a complex group. Note that when working with unitary connections, the natural group to act is $\mathcal{G}$: elements of $\mathcal{G}$ will act by pull-back on connections, while the pull-back of a connection by an element of $\mathcal{G}^{\mathbb{C}}$ may not any more be unitary.

Let us now begin to introduce our parabolic structures. As in the last section, we will work with the vector $\boldsymbol{s}$ of positive real numbers, the n marked points $\{p_1, \ldots, p_n\} \subset \mathbb{CP}^1$ and the parabolic structures at each point p_j with weights $-s_j$ and s_j. What may vary in these bundles – other than the underlying holomorphic structure – is therefore the middle step in the flag at each p_j, a choice of a line in the fiber over p_j. Thus the parabolic bundles of this type are encoded precisely by $\mathcal{C} \times (\mathbb{CP}^1)^n$, and a $g \in \mathcal{G}^{\mathbb{C}}$ acts here in the usual way on the $\mathcal{C}$ factor and on the j^{th} $\mathbb{CP}^1$ by $g\big|_{p_j} \in SL(2, \mathbb{C})$. In other words, $\mathcal{C} \times (\mathbb{CP}^1)^n$ is nothing other than $\mathcal{V}_{\boldsymbol{s}}$ and the map $\overline{\xi}$ is simply inclusion onto the $(\mathbb{CP}^1)^n$ factor, which is certainly holomorphic, as claimed in the last section.

To do gauge theory in these spaces of bundles, as we shall in a moment, it is easier to work with a space of complex structures or connections alone, rather than also to carry along the $(\mathbb{CP}^1)^n$. But $SL(2, \mathbb{C})$ acts transitively on $\mathbb{CP}^1$ and is connected, so there are elements of $\mathcal{G}^{\mathbb{C}}$ which take any point of $\mathcal{C} \times (\mathbb{CP}^1)^n$ to one with all of the $\mathbb{CP}^1$ components equal to some standard basepoint. The stabilizer in $\mathcal{G}^{\mathbb{C}}$ of such bundles with standardized intermediate flag step at all p_j is described by the

Definition 9.3. *The subgroup $\mathcal{G}_P^{\mathbb{C}} = \{g \in \mathcal{G}^{\mathbb{C}} \mid g\big|_{p_j} \in P \ \forall j\} \subset \mathcal{G}^{\mathbb{C}}$, where P is the standard parabolic subgroup of $SL(2, \mathbb{C})$, is the* **asymptotically parabolic complex gauge group**.

We find that $\left(\mathcal{C} \times \mathbb{CP}^1\right)/\mathcal{G}^{\mathbb{C}} \cong \mathcal{C}/\mathcal{G}_P^{\mathbb{C}}$, where $\mathcal{C}$ can now be understood to be the space of parabolic bundles endowed with the same, standard filtration and weights $-s_j$ and s_j at each marked point p_j.

We make one last simplification before introducing the actual spaces of connections and complex structures that we shall need. For this, observe that the existence of local holomorphic frames in holomorphic vector bundles amounts to saying that every complex structure can be gauged trivial in one or even several disjoint disks. In fact, as a constant local complex gauge transformation sends local holomorphic frames to local holomorphic frames, we can even achieve the local trivialization of a $\bar{\partial}_A \in \mathcal{C}$ by a $g \in \mathcal{G}_P^{\mathbb{C}}$ (or, for that matter, by a g which is the identity at each p_j). It thus is appropriate to make the

Definition 9.4. *Let $\mathcal{C}_c$ be the subset of $\mathcal{C}$ of elements which equal $\bar{\partial}$ in a neighborhood of each p_j and $\mathcal{G}_{P,c}^{\mathbb{C}}$ be the subgroup of $\mathcal{G}_P^{\mathbb{C}}$ of automorphisms which are $\bar{\partial}$-holomorphic in some neighborhood of the set $\{p_1, \ldots, p_n\}$.*

(The subscript "c" in $\mathcal{C}_c$ is intended to remind the reader that it consists of operators $\bar{\partial}_A$ for which $\bar{\partial}_A - \bar{\partial}$ is compactly supported in $\mathbb{CP}^1 \setminus \{p_1, \ldots, p_n\}$.) Then $\mathcal{C}/\mathcal{G}_P^{\mathbb{C}} \cong \mathcal{C}_c/\mathcal{G}_{P,c}^{\mathbb{C}}$.

It is still necessary to think of $\mathcal{C}_c$ as a space of parabolic bundles by imposing the weights $\pm s_j$ and standard filtration in the fiber at each p_j. In order to incorporate this external data directly into the complex structures, we imagine removing the marked points from the base $\mathbb{CP}^1$ entirely, and introducing a singularity into the elements of $\mathcal{C}_c$. A direct way to do this is to act on the elements of $\mathcal{C}_c$ by a fixed (very singular, non-unitary) gauge transformation $g_{\boldsymbol{s}}$ which near each marked point p_j looks like $\begin{pmatrix} r_j^{-s_j/2} & 0 \\ 0 & r_j^{s_j/2} \end{pmatrix}$, where $r_j = |z_j|$ for z_j a holomorphic coordinate near p_j. We shall in particular denote by $\bar\partial_X$ the image of $\bar\partial$ under the action of $g_{\boldsymbol{s}}$, and note that near each point p_j

$$\bar\partial_X = \bar\partial + \frac{\mathrm{d}\bar z_j}{2\bar z_j} \otimes \begin{pmatrix} -s_j/2 & 0 \\ 0 & s_j/2 \end{pmatrix}. \tag{9.1}$$

Note that if $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the standard – constant, and thus $\bar\partial$-holomorphic – basis of sections coming from the trivialization of E, then $\{r^{s_j/2}\boldsymbol{e}_1, r^{-s_j/2}\boldsymbol{e}_2\}$ is a local $\bar\partial_X$-holomorphic frame near p_j. We can now make the

Definition 9.5. *Let $\mathcal{C}_{\boldsymbol{s},c}$ be the space of holomorphic structures on the trivial bundle E of rank 2 over $\mathbb{CP}^1 \setminus \{p_1, \dots, p_n\}$ inducing the standard $\bar\partial$ on* det *E and which on some sufficiently small disk around each p_j equal $\bar\partial_X$. Likewise, let $\mathcal{G}^{\mathbb{C}}_{A,c}$ be the group of smooth automorphisms g of E with trivial determinant and for which $g_{\boldsymbol{s}} g {g_{\boldsymbol{s}}}^{-1}$ can be extended by an element of $A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \mid a \in \mathbb{C} \setminus \{0\}\} \subset SL(2,\mathbb{C})$ at p_j to be holomorphic in a neighborhood of p_j. Finally, let $\mathcal{G}_{T,c}$ be the subgroup of unitary elements $\mathcal{G}^{\mathbb{C}}_{A,c}$.*

As our choice of $g_{\boldsymbol{s}}$ is entirely non-canonical – we made no restrictions whatsoever upon its values far away from the marked points – the basepoint $\bar\partial_X$ is somewhat arbitrary and it is appropriate to think of $\mathcal{C}_{\boldsymbol{s},c}$ as merely an affine space.

Since we have punctured the base $\mathbb{CP}^1$, the appropriate Sobolev metric to work with becomes a much more subtle issue. In fact, it is necessary to use Sobolev spaces weighted near each p_j by a power $r_j^{-\delta}$ of the local radial coordinate, where $\delta > 0$ must be chosen smaller than both $\min_j s_j$ and $\min_j \left(1 - s_j\right)$; see [16] for an extensive discussion of these analytic issues. Completing with respect to these norms gives complex structures and gauge transformations which differ from the models in Definition 9.5 by terms which decay rapidly as one approaches each point p_j.

Definition 9.6. *We shall denote by $\mathcal{C}_{\boldsymbol{s},\delta}$ the completion of $\mathcal{C}_{\boldsymbol{s},c}$ in the δ-weighted Sobolev L^2_1 norm and by $\mathcal{G}^{\mathbb{C}}_{A,\delta}$ and $\mathcal{G}_{T,\delta}$ the completions of $\mathcal{G}^{\mathbb{C}}_{A,c}$ and $\mathcal{G}_{T,c}$ with respect to the $L^2_{2,\delta}$ norm.*

A first application of analysis in these weighted Sobolev spaces shows that every $\mathcal{G}^{\mathbb{C}}_{A,\delta}$-orbit in $\mathcal{C}_{\boldsymbol{s},\delta}$ contains elements that look like (9.1) near each p_j, see [16]. This is a sort of Newlander–Nirenberg theorem, which is normally trivial over a manifold of complex dimension one, but here is rendered difficult again by the singularities. It provides an important step in proving the

Proposition 9.1. *There exists an identification $\mathcal{C}_c/\mathcal{G}^{\mathbb{C}}_{P,c} = \mathcal{C}_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$.*

Proof. Conjugation by $g_{\boldsymbol{s}}$ gives a well-defined map $\overline{\sigma} : \mathcal{C}_c \to \mathcal{C}_{\boldsymbol{s},c} \hookrightarrow \mathcal{C}_{\boldsymbol{s},\delta}$; certainly $\bar\partial_X \in \mathcal{C}_{\boldsymbol{s},c}$. If $\bar\partial_A, \bar\partial_B \in \mathcal{C}_c$ are equivalent by some $g \in \mathcal{G}^{\mathbb{C}}_{P,c}$, then $\overline{\sigma}(\bar\partial_A)$ and $\overline{\sigma}(\bar\partial_B)$ will be equivalent by ${g_{\boldsymbol{s}}}^{-1} g g_{\boldsymbol{s}}$. Say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix of holomorphic functions defining g near p_j, with therefore $c(p_j) = 0$ and $a(p_j) = d(p_j)^{-1}$. Then ${g_{\boldsymbol{s}}}^{-1} g g_{\boldsymbol{s}}$ will look like $\begin{pmatrix} a & br^{s_j} \\ cr^{-s_j} & d \end{pmatrix}$ near p_j. But br^{s_j} decays at least like r^{s_j} towards p_j and cr^{-s_j} at least like r^{1-s_j}, both of which are in the weighted Sobolev closure we are using, so ${g_{\boldsymbol{s}}}^{-1} g g_{\boldsymbol{s}} \in \mathcal{G}^{\mathbb{C}}_{A,\delta}$.

Hence $\overline{\sigma}$ induces a well-defined map $\sigma : \mathcal{C}_c/\mathcal{G}^{\mathbb{C}}_{P,c} \to \mathcal{C}_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$. The singular Newlander-Nirenberg theorem mentioned above tells us that σ is surjective, so it only remains to show that it is injective. So say that $\overline{\sigma}(\bar\partial_A)$ and $\overline{\sigma}(\bar\partial_B)$ are equivalent by some element $h \in \mathcal{G}^{\mathbb{C}}_{A,\delta}$. Concentrating near one p_j, let $h = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} + h^j$, where h^j decays towards p_j at least like r^δ, a is holomorphic near p_j and $a(p_j) \neq 0$. But then $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} + g_{\boldsymbol{s}} h^j g_{\boldsymbol{s}}{}^{-1}$ takes $\bar\partial_A$ to $\bar\partial_B$, and thus must be holomorphic in a punctured neighborhood of p_j. If $\begin{pmatrix} h^j_{11} & r^{-s_j} h^j_{12} \\ r^{s_j} h^j_{21} & h^j_{22} \end{pmatrix}$ is to be holomorphic and h^j is to decay, then h^j_{11} and h^j_{22} must be have a zero at p_j while h^j_{12} and h^j_{21} must be of the form br^{s_j} and cr^{-s_j}, respectively, for b and c holomorphic functions on a neighborhood of p_j. However if h^j is to decay at p_j then $c(p_j)$ must equal zero while $b(c_j)$ can be any complex number – in other words, $g_{\boldsymbol{s}} h g_{\boldsymbol{s}}{}^{-1} \in \mathcal{G}^{\mathbb{C}}_{P,\,c}$ and we are done. □

10. The Second Kempf–Ness-Type Theorem

We now have the complex group $\mathcal{G}^{\mathbb{C}}_{A,\delta}$ acting on the complex affine space $\mathcal{C}_{\boldsymbol{s},\delta}$ preserving its complex structure. $\mathcal{G}^{\mathbb{C}}_{A,\delta}$ is the complexification of the group $\mathcal{G}_{T,\,\delta}$, which will have to be the analogue in this infinite-dimensional situation of the compact group acting symplectically. It does indeed preserve the constant symplectic form

$$\omega(u, v) = \int_{\mathbb{CP}^1} u \wedge v,$$

where u and v are tangent vectors to $\mathcal{C}_{\boldsymbol{s},\delta}$ and the operation $u \wedge v$ takes the normal wedge of the form parts and the Killing form on the Lie algebra parts. In fact, Atiyah and Bott define and study this symplectic form in [1] – for higher genus and without parabolic structures, but an identical argument shows that the moment map for the action of $\mathcal{G}^{\mathbb{C}}_{A,\delta}$ is nothing other than the map which assigns to a connection d_A its curvature F_A.

Hence one would now hope – if the appropriate infinite-dimensional Kempf–Ness theorem could be proven – that the quotient $\mathcal{C}^s_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$ could be identified with the symplectic quotient $\mathcal{C}^s_{\boldsymbol{s},\delta}/\!\!/\mathcal{G}_{T,\,\delta} = \mathcal{F}^{\mathrm{irr}}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\,\delta}$, where $\mathcal{F}_{\boldsymbol{s},\delta}$ is the subset of flat connections (the zero set of the current moment map) in $\mathcal{C}_{\boldsymbol{s},\delta}$ while $\mathcal{F}^{\mathrm{irr}}_{\boldsymbol{s},\delta}$ denotes the flat and irreducible connections. This was indeed done from this perspective for bundles without parabolic structure by Donaldson in [5] and by several authors for parabolic bundles, after the initial, purely algebro-geometric proof of Mehta and Seshadri in [15]. None of these proofs applies directly to our present situation, requiring either all of the parabolic weights to lie in $[0, 1)$, there to be only one parabolic point, the base Riemann surface Σ to have genus two or greater, or several of these conditions. (But note that an analogue of the Mehta–Seshadri theorem which would suffice for our purposes is proven in the preprint [2].) Let us instead sketch a proof which applies here, based on [16] and quoting some facts that are now known about the topology of the space of parabolic bundles.

First of all, the topology of $\mathcal{V}^s_{\boldsymbol{s}}/\,\mathrm{Iso}^0$ has been extensively studied, and in particular it is known to be connected, see [8,3] or [9]. But $\mathcal{V}^s_{\boldsymbol{s}}/\,\mathrm{Iso}^0 = \mathcal{C}^s_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$ and $\mathcal{G}^{\mathbb{C}}_{A,\delta}$ is connected (since $\pi_2(SL(2,\mathbb{C}))$ is trivial), so it follows that the set $\mathcal{C}^s_{\boldsymbol{s},\delta}$ of stable connections is connected. Now an implicit function theorem argument shows that $\mathcal{G}^{\mathbb{C}}_{A,\delta} \cdot \mathcal{F}^{\mathrm{irr}}_{\boldsymbol{s},\delta}$ is open in $\mathcal{C}^s_{\boldsymbol{s},\delta}$ and a version of Uhlenbeck compactness in our weighted spaces shows that it is also closed. (See [16] for the analytic details.) Thus every stable connection is complex gauge-equivalent to a (unitary gauge orbit of a) flat connection; we will write κ_2 for the resulting diffeomorphism of $\mathcal{C}^s_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta}$ with $\mathcal{F}^{\mathrm{irr}}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\,\delta}$. In fact, [16] also shows that κ_2

can be extended to a homeomorphism $\kappa_2 : \mathcal{V}_{\boldsymbol{s}}^{ss}/\!\sim\, \to \mathcal{F}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\delta}$, just as at the end of Sect. 8.

11. Representations of the Fundamental Group

It is by now a classic result of modern differential geometry that the moduli space of flat connections on some manifold M can be identified with the moduli space of representations of $\pi_1(M)$, by associating to a flat connection d_A its holonomy representation $\overline{\mathrm{hol}}(\mathrm{d}_A)$. In our case, the image of $\overline{\mathrm{hol}}$ will not be all representations of $\pi_1(\mathbb{CP}^1 \smallsetminus \{p_1, \ldots, p_n\})$ since we are not working with all flat connections but only those which are in $\mathcal{F}_{\boldsymbol{s},\delta}$. These are all connections which differ from the model (the unitary connection corresponding to the basepoint $\bar{\partial}_X \in \mathcal{C}_{\boldsymbol{s},\delta}$) by terms which decay fast enough towards the puncture that their holonomy along a small loop around each p_j will be the same as the local model, i.e., conjugate to $\begin{pmatrix} e^{-\pi i s_j} & 0 \\ 0 & e^{\pi i s_j} \end{pmatrix}$; see [16] for details.

Let us give these representations a name.

Definition 11.1. *With* s *fixed as usual, the* **restricted representation variety** $\mathcal{R}_{\boldsymbol{s}}$ **of** $\mathbb{CP}^1 \smallsetminus \{p_1, \ldots, p_n\}$ *consists of those homomorphisms from* $\pi_1(\mathbb{CP}^1 \smallsetminus \{p_1, \ldots, p_n\})$ *to* $SU(2)$ *which take a small loop around each* p_j *to elements of trace* $2\cos(\pi s_j)$*;* $\mathcal{R}_{\boldsymbol{s}}$ *admits an action of* $SU(2)$ *by post-conjugation and the quotient* $\mathcal{R}_{\boldsymbol{s}}/SU(2)$ *shall be called the* **moduli space of restricted representations**. *We shall also denote by* $\mathcal{R}_{\boldsymbol{s}}^{\mathrm{irr}}$ *the subset of irreducible elements in* $\mathcal{R}_{\boldsymbol{s}}$.

With this understood, we can say that the holonomy representation $\overline{\mathrm{hol}}$ induces a map $\mathrm{hol}\colon \mathcal{F}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\delta} \to \mathcal{R}_{\boldsymbol{s}}/SU(2)$ which, it is easy to see, is smooth on the smooth part $\mathcal{F}_{\boldsymbol{s},\delta}^{\mathrm{irr}}/\mathcal{G}_{T,\delta}$.

In order to show that hol is a bijection, we should construct a flat connection $\mathrm{d}_A \in \mathcal{F}_{\boldsymbol{s},\delta}$ whose holonomy representation is an arbitrary fixed $\rho \in \mathcal{R}_{\boldsymbol{s}}$. The usual approach to this is to write $\mathbb{CP}^1 \smallsetminus \{p_1, \ldots, p_n\} = \mathcal{H}^2_{\mathbb{R}}/\Gamma$, where $\pi(\mathbb{CP}^1 \smallsetminus \{p_1, \ldots, p_n\}) \xrightarrow{\cong} \Gamma \subset PSL(2, \mathbb{R})$ and to consider the connection d_ρ induced on $\mathcal{H}^2_{\mathbb{R}} \times_\rho \mathbb{C}^2$ by the trivial connection on $\mathcal{H}^2_{\mathbb{R}} \times \mathbb{C}^2$. But if we choose coordinates $\{x_j, y_j\}$ near the cusp corresponding to the parabolic element $\gamma_j \in \Gamma$ of a fundamental domain for the Γ-action on $\mathcal{H}^2_{\mathbb{R}}$ which look like an infinite vertical strip in the upper half-plane of width 2π, as is often done, then d_ρ exactly equals

$$\mathrm{d}_{A_j} = \mathrm{d} - dx_j \otimes C_j^{-1} \begin{pmatrix} -is_j/2 & 0 \\ 0 & is_j/2 \end{pmatrix} C_j$$

on a neighborhood of that cusp, where $\rho(\gamma_j) = C_j^{-1} \begin{pmatrix} e^{-\pi i s_j} & 0 \\ 0 & e^{\pi i s_j} \end{pmatrix} C_j$. If $g : \mathcal{H}^2_{\mathbb{R}}/\Gamma \to SU(2)$ equals C_j near the j^{th} cusp, then $g(\mathrm{d}_\rho)$ is the required flat connection in $\mathcal{F}_{\boldsymbol{s},\delta}$.

It shall be convenient in just a moment to have $\mathcal{R}_{\boldsymbol{s}}$ in another form. Taking the obvious presentation of the fundamental group of a punctured sphere, we can think of a restricted representation as nothing other than a choice of n elements of $SU(2)$, the j^{th} from the conjugacy class of trace $2\cos(\pi s_j)$, such that the product of these elements is the identity, i.e.:

$$\mathcal{R}_{\boldsymbol{s}} = \Big\{(g_1, \ldots, g_n) \in (SU(2))^n \mid g_1 \cdots g_n = \mathrm{Id} \text{ and } \operatorname{Tr} g_j = 2\cos(\pi s_j)\ \forall j\Big\}. \tag{11.1}$$

Furthermore, the reducible representations $\mathcal{R}_{\mathbf{s}} \setminus \mathcal{R}_{\mathbf{s}}^{\text{irr}}$ are those which are simultaneously diagonalizable, so in our present formulation of $\mathcal{R}_{\mathbf{s}}$ they correspond to n-tuples $(g_1, \ldots, g_n)$ such that all

$$g_j = X^{-1} \begin{pmatrix} e^{-\pi i s_j} & 0 \\ 0 & e^{\pi i s_j} \end{pmatrix} X \tag{11.2}$$

with the same X for all j.

12. Returning to Polygons: A Second Gauss Map and Polygons in S^3

The conjugacy classes appearing in (11.1) are simply spheres centered at the identity of radius πs_j in the spherical metric on $SU(2) \cong S^3$. Since $\sum_{j=1}^{n} s_j < 1$ and the s_j are all positive, it follows that each $s_j < 1$, and hence there is a unique directed geodesic ϵ_j from the identity of $SU(2)$ to any of the g_j as in (11.1). Thus $e_j = h\epsilon_j$ is the unique directed geodesic segment from h to hg_j and as g_j runs over all elements of trace $2\cos(\pi s_j)$, e_j exhausts the set of directed geodesic segments starting at h and of length πs_j. This motivates us to recall the definitions of Sect. 2, generalized to a slightly different ambient space:

Definition 12.1. *Let $\mathbf{s}$ be our usual n-tuple of positive real numbers.*

- *The* **configuration space** $\mathcal{P}_{\pi\mathbf{s}}^{S^3}$ **of polygons in** S^3 **with fixed side lengths** $\pi\mathbf{s}$ *is the set of all n-tuples $(e_1, \ldots, e_n)$ of directed geodesic segments such that the length of e_j is πs_j and the endpoint of e_j is the beginning point of $e_{j+1(\text{mod } n)}$. The* **moduli space of polygons** *is then simply $\mathcal{P}_{\pi\mathbf{s}}^{S^3}/SO(4)$ where $SO(4)$ acts diagonally on the n-tuples of geodesic segments.*
- *A polygon is said to be* **degenerate** *if it lies entirely in some great circle in S^3. The set of non-degenerate polygons shall be denoted $\widehat{\mathcal{P}}_{\pi\mathbf{s}}^{S^3}$.*
- *The set of* **based polygons** *is $\mathcal{P}_{\pi\mathbf{s},0}^{S^3} = \{(e_1, \ldots, e_n) \in \mathcal{P}_{\pi\mathbf{s}}^{S^3} \mid e_1^{\text{begin}} = \mathrm{Id} \in SU(2) \cong S^3\}$.*

Hence for a $(g_1, \ldots, g_n) \in \mathcal{R}_{\mathbf{s}}$ we can construct an element $(e_1, \ldots, e_n) \in \mathcal{P}_{\pi\mathbf{s},0}^{S^3}$ by defining $g_0 = \mathrm{Id}$ and then letting e_j be the unique directed geodesic segment from $g_0 \cdots g_{j-1}$ to $g_0 \cdots g_j$ for $1 \le j \le n$. Note that if we start with a reducible representation satisfying (11.2), the corresponding polygon will lie entirely on the great circle

$$\left\{ X^{-1} \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix} X \;\middle|\; t \in [0, 2\pi] \right\}$$

and thus be degenerate. As these constructions are clearly invertible and continuous, it follows that we have a homeomorphism $\overline{G}_2^{\,-1} : \mathcal{R}_{\mathbf{s}} \to \mathcal{P}_{\pi\mathbf{s}}^{S^3}$ which restricts to a diffeomorphism of $\mathcal{R}_{\mathbf{s}}^{\text{irr}}$ with $\widehat{\mathcal{P}}_{\pi\mathbf{s},0}^{S^3}$; we use this notation $\overline{G}_2^{\,-1}$ since the present map is something of an inverse to a non-linear generalization of the first Gauss map we defined in Sect. 3.

Finally, recall that the adjoint group of $SU(2)$ is $SO(3)$. Hence the conjugation action of $SU(2)$ on the matrices encoding a representation corresponds to the residual action of the group $SO(3)$ of isometries of S^3 fixing the identity, which is the basepoint of our based polygons. Thus we can finish our grand tour of moduli spaces by defining the induced homeomorphism $\mathcal{P}_{\pi\mathbf{s}}^{S^3}/SO(4) \cong \mathcal{P}_{\pi\mathbf{s},0}^{S^3}/SO(3) \xrightarrow{G_2} \mathcal{R}_{\mathbf{s}}/SU(2)$ which, as usual, restricts to a diffeomorphism of $\widehat{\mathcal{P}}_{\pi\mathbf{s}}^{S^3}/SO(4)$ with $\mathcal{R}_{\mathbf{s}}^{\text{irr}}/SU(2)$.

13. Conclusion

Let us return to the diagram we had originally intended to fill in, for completeness now adding the various intermediate spaces and maps we used:

$$\begin{tikzcd}
\widehat{\mathcal{P}}^{\mathbb{R}^3}_{\boldsymbol{s}}/\mathbb{E}^+(3) \arrow[d, equal] & & \widehat{\mathcal{P}}^{S^3}_{\pi\boldsymbol{s}}/SO(4) \arrow[d, equal] \\
\widehat{\mathcal{P}}^{\mathbb{R}^3}_{\boldsymbol{s},0}/SO(3) \arrow[d, "G_1"'] & & \widehat{\mathcal{P}}^{S^3}_{\pi\boldsymbol{s},0}/SO(3) \arrow[d, "G_2"'] \\
\widehat{(S^2)^n_{\boldsymbol{s}}} /\!\!/ SO(3) \arrow[d, "\kappa_1"'] & & \mathcal{R}^{\mathrm{irr}}_{\boldsymbol{s}}/SU(2) \\
\mathcal{W}^s_{\boldsymbol{s}}/PSL(2,\mathbb{C}) \arrow[d, "\xi"'] & & \mathcal{F}^{\mathrm{irr}}_{\boldsymbol{s},\delta}/\mathcal{G}_{T,\delta} \arrow[u, "hol"'] \\
\mathcal{V}^s_{\boldsymbol{s}}/\,\mathrm{Iso}^0 \arrow[d, equal] & & \mathcal{C}^s_{\boldsymbol{s},\delta}/\mathcal{G}^{\mathbb{C}}_{A,\delta} \arrow[u, "\kappa_2"'] \\
\left(\mathcal{C}\times(\mathbb{CP}^1)^n\right)^s/\mathcal{G}^{\mathbb{C}} \arrow[r, equal] & \mathcal{C}^s/\mathcal{G}^{\mathbb{C}}_P \arrow[r, equal] & \mathcal{C}^s_c/\mathcal{G}^{\mathbb{C}}_{P,c} \arrow[u, "\sigma"']
\end{tikzcd}$$

Theorem 13.1. *In the above diagram, all maps are diffeomorphisms and those connecting complex spaces are biholomorphisms. All maps also extend to homeomorphisms on the corresponding spaces of semi-stable points.*

Let us conclude by mentioning two other maps which could be inserted in the above diagram. The first is a diffeomorphism $\widehat{\mathcal{P}}^{S^3}_{\pi\boldsymbol{s},0}/SO(3) \to \widehat{\mathcal{P}}^{\mathbb{R}^3}_{\boldsymbol{s},0}/SO(3)$ defined by M. Sargent in [17], as follows. On the level of configuration spaces of based polygons, he enlarges the S^3 by a factor $k \to \infty$, all the while scaling the polygons by $1/k$, getting in the limit a polygon with the original side lengths but on $\mathbb{R}^3$. As the resulting map on configuration spaces is equivariant with respect to the $SO(3)$-action on both sides, a diffeomorphism of the moduli spaces results. This construction is admirably direct, but has the disadvantage that it does not reveal whether any geometric structures are preserved.

A more sophisticated approach is taken by L. Jeffrey in [10] with a map

$$\widehat{(S^2)^n_{\boldsymbol{s}}} /\!\!/ SO(3) \to \mathcal{R}^{\mathrm{irr}}_{\boldsymbol{s}}/SU(2).$$

For more general compact groups – not just $SU(2)$ – she gives a symplectomorphism between the quotient of a submanifold of the product of conjugacy classes in the Lie algebra and the corresponding quotient in the group itself. Her main interest is symplectic structures, however, and not the complex structures we address here; nor does she use or address the configuration spaces of weighted points and parabolic bundle techniques which form the core of Sects. 5–10, above.

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