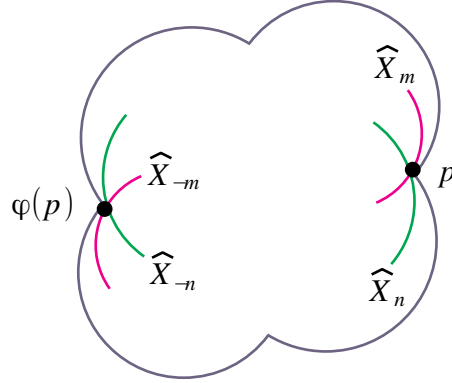


FIGURE 4.1. $\widehat{X}_n^\partial = \widehat{X}_m^\partial$ is a single point

We now turn our attention to tangency of X_n 's. This turns out only to be an interesting issue for loxodromic generators γ , so we shall assume we are in that case for the remainder of this section. Further, since all $\widehat{X}_n$'s are centered on $\widehat{\mathbb{C}}$, any point of tangency will also lie on $\widehat{\mathbb{C}}$. So we shall also now restrict our attention to $\widehat{X}_n$'s and $\widehat{\Delta}$, objects which lie on $\widehat{\mathbb{C}}$.

Our conventions for loxodromic γ imply that 1 is in the interior and 0 the exterior of $\widehat{X}_n$ when $n > 0$. Thus if $\widehat{X}_n$ and $\widehat{X}_m$ are externally tangent, then n and m must be of different signs. If $\widehat{X}_n$ is internally tangent to $\widehat{X}_m$, then n and m are of the same sign and $|n| > |m|$.

It is convenient to make the following

Definition 4.6. A point $p \in \partial\widehat{\Delta}$ is a *smooth point of the boundary* if p lies in the interior of a side.

It is possible for a smooth point p of the boundary to be in the interior of two sides S_n and S_m which are externally tangent at p . It is also possible that p is a corner and a smooth point of the boundary, such as when there exists an X_n which is internally tangent to a side at p .

Proposition 4.7. *Suppose γ is loxodromic.*

- (a) *If $\widehat{X}_m$ is internally tangent to S_n at a smooth point p and $m > n > 0$, then $m = 2n$.*
- (b) *If S_k is externally tangent to S_n at a smooth point q , then $k = -n$ and $q = \frac{1}{2} \in \widehat{\mathbb{C}}$.*

Furthermore, if either one of these cases occurs, then the other one also occurs and $\gamma^{-n}p = q$.

Proof. Suppose that $m > n > 0$ and $\widehat{X}_m$ is internally tangent to the side S_n at a smooth point p .

By Proposition 4.3, $\widehat{X}_{-m}$ is internally tangent to the side S_{-n} at the smooth point t which, by Proposition 4.2 must be the point $\gamma^{-m}p$. Further, $q = \gamma^{-n}p$ is a smooth point on S_{-n} . $\gamma^{m-n}t = q$ and by Proposition 4.1 this implies that $q \in \widehat{X}_{m-n}$ and $t \in \widehat{X}_{n-m}$. Since t is a smooth point of S_{-n} , either $\widehat{X}_{n-m}$ contains or is internally tangent to S_{-n} ; see Figure 4.2. The latter is impossible, since either