PERELMAN PROVES POINCARÉ

JONATHAN A. PORITZ

ABSTRACT.
In 1904, **Henri Poincaré** conjectured:

Every compact, connected, simply-connected 3*-manifold is the* 3*-sphere.*

This conjecture has been near the center of a maelstrom of activity in topology, geometry, analysis, and many allied and sub-disciplines for a hundred years. Recently, Grisha Perelman announced a proof of the conjecture – in fact of the stronger **Geometrization Conjecture** of **William Thurston** (from 1982), which states:

Every compact, connected 3*-manifold can be cut along embedded spheres and tori into pieces each of which has a geometric structure from the Eightfold Path.*

[See below for more (and more meaningful) details).]

In this paper I collect many of the definitions and a few of the theorems which an interested mathematical tourist will need to understand the outlines, at the least, of the progress made in this area. The goal here is to have fairly precise and accurate definitions (and pointers to some literature) which would help someone who became deeply interested in this area to begin the lifetime of study that would inevitably ensue.

-1. A DREAM (A NIGHTMARE?)

You are in a dark room in a shabby hotel. You hear someone come into the room. They hand you a strange object and demand, in a heavy French accent, to know what it is. You feel it carefully, trying to understand. You can certainly feel that it is three-dimensional [*it's a* 3*-manifold*]. You pinch one corner and hold it up, no pieces fall off [*it's connected*]. You cannot feel any sharp edges, nor does it trail out the door or window [*it's compact*]. You think of hanging it from a hook in the ceiling so as to get a better gander at it with both hands, but it is very squishy and flexible so you cannot tie a string merely around a protrusion [*we are interested in identification up to homeomorphism or diffeomorphism (or PL-diffeomorphism), which are rather flexible notions, not at all rigid like an isometry would have to be*], nor can you find a part of the object through which to thread your string – any loop of string you put on the object shrinks to nothing and the object falls to the floor [*it's simply-connected*].

The walls of the hotel are atrociously thin. Next door, you hear the hotel guest (whose name, you noticed in the ledger, is Stephen Smale) whistling "The Girl from Ipanema" and laughing about how easy it is to untangle highly twisted disks when you can untwist in many dimensions. He seems to be saying he settled with the Frenchman. [*Smale, and, shortly after, Stallings and Wallace, proved the Poincaré conjecture in dimensions* 5 *and higher.*]

On the other side, you hear very strange sounds, as though someone were assembling a chopped-liver swan. Another guest (Michael Freedman – you also noticed *his* name) seems to be working very hard, and to great effect, on very squishy objects. [*Freedman proved the* 4*-dimensional topological Poincaré Conjecture, as part of a complete classification of the homeomorphism types of* 4*-manifolds.*] A British accented voice in that room shouts "What about this!?!", whereafter there is great commotion and then relative silence. [*Simon Donaldson proved the existence of a* $\mathbb{R}^4_{fake}$ *which is homeomorphic, but not diffeomorphic, to the usual* $\mathbb{R}^4$*; Gompff and others then found many more such strange examples, but their complete classification is still mysterious.*]

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Someone new comes into your room, smelling of surfboard wax and talking about "going with the flow". He seems to have a good idea what your mystery object is, but he gets nervous when you tell him you weren't very positive about the object from the beginning. [*Richard Hamilton used the Ricci flow to prove important results in dimension three, but his strongest results were in the case of manifolds of everywhere positive Ricci curvature.*] He retreats to the corner of your room, muttering about cigars and pencil-neck geeks. [*Hamilton's progress was stopped by worries about a stable (and useless) 'cigar soliton' or thin necks appearing in finite time in the Ricci flow.*]

A noisy crowd fills the hallway for a while, you hear them shouting for someone named "Wild Bill". This Bill may be a Buddhist (or just drunk), because they are all talking loudly about an "eightfold way", and he is boasting that the Frenchman is completely taken care of by one of these eight ways. [*William Thurston generalized the Poincaré conjecture to a proposed classification of all* 3*-manifolds – called the "Geometrization Conjecture" – consisting of a decomposition of these manifolds into canonical simple pieces, each of which must then admit one of eight particular geometries; from this would follow Poincaré without trouble. Thurston advanced this theory on many fronts, but seemed to have no approach for the complete proof.*]

After what seems like a hundred years, a very quiet Russian enters your room. He has Bill in tow, and he reassures the fearful surfer that everything is OK. Standing (gently) on the surfer's shoulders, he repairs the light fixture in your room, and you now have enough illumination [*thanks to people like Morgan, Tian, Kleiner and Lott*] to see that your mystery does indeed flow, that it has no cigars and the necks it does develop are not to be feared – in fact, the necks point right at Bill's eightfold way. That damn Frenchman is gone but your mystery object is just a 3-sphere. [*Grisha Perelman, in a few short papers – whose complete details have since been filled in by careful work of Morgan et al. – made the Ricci flow with surgery work. It appears that the complete Geometrization Conjecture, and particularly the Poincaré Conjecture, is now settled in the affirmative.*]

0. Introduction

Henri Poincaré was one of the last great mathematical polymaths (along with the other great name in European mathematics of his day, David Hilbert), making fundamental contributions to algebraic topology and dynamical systems, as well as various parts of mathematical physics and even the philosophy of science. In a paper [Poi00] from 1900, he conjectured that the only compact, connected 3-manifold which had the Betti numbers of the 3-sphere S^3 was in fact S^3 itself. However, he found a counterexample in 1904, in [Poi04], where he also proposed a new criterion for recognition of S^3: that the manifold have the same homotopy groups as S^3; it is this version which came to be called *The Poincaré Conjecture*.

A number of incorrect proofs of the Poincaré Conjecture have been announced over the years, some making it into print in quite reputable journals. Probably this is because the Conjecture is so simple to state, like another famous old problem which was solved a few years ago. But, unlike Fermat's Last Theorem, whose statement can be well understood by anyone with a high-school mathematics education, the Poincaré Conjecture is stated with words which are not necessarily in the vocabulary even of advanced undergraduate mathematics majors.

Even though understanding of the statement of the Poincaré Conjecture is not widespread, this corner of pure mathematics has received an enormous amount of attention in the last few years. Nearly every major news outlet in the industrialized world had at least a passing reference to the solution of this one hundred year-old problem, moreover by a modest, unassuming, and young Russian mathematician named Grisha Perelman who then refused the Fields medal and all attention. Further fanning the flames was the contrast between Perelman's modesty and the aggressive drive to claim partial priority in this work, by some individuals.

References 1. Typical of the popular press's approach to Perelman's involvement in the resolution of the Poincaré Conjecture are [Cha] and [Kes06] (and nearly countless similar others); the article that discussed (and helped to publicize) the priority dispute was [NG].

Passing from the personal to the mathematical, the goal of this paper is to put down in one place definitions of all of the technical terms which appear in the Poincaré Conjecture, assuming as background something along the lines of advanced calculus and some of both linear and abstract algebra. What takes more time and care is setting the Conjecture in some kind of context, which I try to accomplish by some discussion, a liberal sprinkling of further definitions, along with examples, facts [theorems], and exercises – also a number of pointers to the vast background literature in this area are provided. Even more layers must be investigated in order to set up a few of the ideas of the Conjecture's recent proof; in particular, the proof actually goes as far as proving William Thurston's amazing *Geometrization Conjecture*, so I include enough to make a careful statement of that Conjecture as well.

The following sections §§1-5 lay out various background. Then in §6 the case of two-dimensional manifolds is presented as an example of the type of result, and the strategy for its proof, which will then be pursued for dimension three in §7. That section also carefully states both the Geometrization and Poincaré Conjectures and provides a very, very brief sketch of Perelman's proofs. Finally, §8 describes some of the work on the Generalized Poincaré Conjecture in higher dimensions.

First we need a few preliminaries:

Definition 0.1. I use the standard notation for things like $\mathbb{N}$ and $\mathbb{R}$ and $\mathbb{C}$, but perhaps I should mention:

$$\begin{array}{rl} \text{for } n \in \mathbb{N} \cup \{0\}, & S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\} \\ & C^k = \{\text{functions (on an appropriate domain) which have } k \text{ continuous derivatives}\} \\ & C^\infty = \{\text{“smooth” functions}\} = \cap_{k=1}^\infty C^k \\ & F_{abc\ldots} = \text{the free group on generators } a, b, c, \ldots \ (\textit{i.e.,} \text{ those generators and no relations}) \\ \text{for } \textit{operators } A \text{ and } B, & [A,B] = A \circ B - B \circ A \ \ \text{(but beware similar notation in homotopy theory, see §2, below)} \end{array}$$

Definition 0.2. Group actions:

- We say that a group G *acts on* a space X if there is a map $G \times X \to X : (g, x) \mapsto g \cdot x$ satisfying $(gh) \cdot x = g \cdot (h \cdot x)\ \forall g, h \in G, x \in X$ and $id_G \cdot x = x\ \forall x \in X$.
- The *orbit* $G \cdot x$ of $x \in X$ is the set $\{g \cdot x \mid g \in G\}$, and if any orbit is all of X we say that the action is *transitive*.
- The *stabilizer* of a point $x \in X$ is the subgroup $G_x = \{g \in G \mid g \cdot x = x\} \leq G$ and we say that the action is *free* if all stabilizers are trivial.
- The *quotient space* X/G is the collection of equivalence classes of points in X, where two points are equivalent if there is some group element in G which maps one to the other, and comes with a surjective *projection map* $p_{X/G} : X \to X/G$ which sends a point $x \in X$ to its equivalence class.

1. Point-Set Topology

Definition 1.1. A *topological space* is a set X together with a collection $\mathcal{O}$ of subsets of X satisfying: *(i)* $\emptyset, X \in \mathcal{O}$; *(ii)* arbitrary unions of elements in $\mathcal{O}$ are again in $\mathcal{O}$; and *(iii)* finite intersections of elements in $\mathcal{O}$ are again in $\mathcal{O}$. Elements of $\mathcal{O}$ are called *open sets*, while complements of open sets are called *closed sets.*

The idea of a topology is that these open sets are *neighborhoods* of each of their points; that is, the collection of all open sets O_x containing some point $x \in X$ is the collection of all “sufficiently small neighborhoods of x”=“sets of points sufficiently close to x”, however *in a way that doesn't require a definition of distance.* Hence it makes sense to define

Definition 1.2. A *continuous map* $f : X \to Y$ between topological spaces is a map for which the inverse image of any open set (in Y) is open (in X).

Many topologies are *metrizable*, in that there is a metric (in the sense, at the moment, of a function which measures the distances between pairs of points) for which the open sets are simply unions of open balls – where an open ball in a metric space is the collection of those points closer to some fixed point than a given (positive real) number. All topological spaces I shall deal with in this paper are metrizable.

Definition 1.3. A nonempty subset A of a topological space $(X, \mathcal{O})$ is said to be a *connected component* if it is both open and closed. A topological space which consists of only one connected component is said to be *connected.*

Definition 1.4. A subset K of a topological space $(X, \mathcal{O})$ is said to be *compact* if any open cover of K (a collection of open sets whose union contains K) has a finite subcover. For metrizable spaces, this is equivalent to: any infinite sequence of points of K has a convergent subsequence.

Definition 1.5. A *homeomorphism* between topological spaces is a continuous bijection with continuous inverse. Two spaces are *homeomorphic* if there exists a homeomorphism between them.

Exercise 1.6. Find topologies $\mathcal{O}_1$ and $\mathcal{O}_2$ on the set $\mathbb{R}$ such that the only continuous functions from $(\mathbb{R}, \mathcal{O}_1)$ to $(\mathbb{R}, \mathcal{O}_2)$ are constant. Find other topologies so that **all** functions are continuous. [*Hint:* try very "coarse" or "fine" topologies – that is, ones with very few or very many open sets.]

Exercise 1.7. Fix $n \in \mathbb{N}$. On $\mathbb{C}^n$ consider the collection $\mathcal{C}$ of sets which are the zero sets of complex polynomials in n variables. Does this define the closed sets of some topology on $\mathbb{C}^n$ – *i.e.,* is $\mathcal{O} = \{A^c \mid A \in \mathcal{C}\}$ a topology? [*Hint:* if it were a topology, it would be called the *Zariski topology.*] On $\mathbb{C}$ or $\mathbb{C}^2$, for example, does this topology agree with the usual one? If so, exhibit a homeomorphism; if not, show some qualitatively different behavior.

Exercise 1.8. *EXTRA CREDIT:* Show that $\mathbb{R}^n$ is never homeomorphic to $\mathbb{R}^m$ if $n \neq m$. Is there ever a bijection between Euclidean spaces of different dimensions? [*Research:* "space-filling curves".]

References 2. A very gentle introduction to some of this basic topology (along with some geometry we will see in later sections of this paper) is [ST67]. There are also many standard textbooks in this area, such as, for example, [Mun99] and [Cro05].

Before we move on, let us put together a little topology and algebra.

Definition 1.9. A *topological group* is a group G which is also a topological space and for which the maps $G \times G \to G : (g, h) \mapsto g \cdot h$ and $G \to G : g \mapsto g^{-1}$ are continuous. When a topological group G acts on a topological space $(X, \mathcal{O}_X)$, I shall always assume that the map $G \times X \to X$ is continuous. The quotient space X/G can then be given the *quotient topology* whose open sets are $\mathcal{O}_{X/G} = \left\{ p_{X/G}^{-1}(U) \mid U \in \mathcal{O}_X \right\}$.

2. Homotopy

Definition 2.1. Two continuous maps between topological spaces $f_0, f_1 : X \to Y$ are said to be *homotopic*, written $f_0 \simeq f_1$, if there exists a continuous map $F : [0,1] \times X \to Y$ such that $F(0,t) = f_0(t)$ and $F(1,t) = f_1(t)$. The collection of homotopy classes of maps from X to Y is written $[X, Y]$. [Note that "[0,1]" still means the closed interval of real numbers from 0 to 1, but that should not be confused with the homotopy theory use of this symbol.]

Definition 2.2. For a topological space X, $\pi_n(X) = [S^n, X]$.

If $n = 0$: $\pi_0(X)$ is the collection of *path components* of X. Usually – certainly for all the spaces I shall deal with in this paper – this is the same as the set of connected components of X.

If $n > 0$: $\pi_n(X)$ is a group under *concatenation* of paths (*i.e.*, follow one path then the other), and therefore $\pi_n(X)$ is also called the *nth homotopy group of* X. A special case:

Definition 2.3. The *fundamental group* of a topological space X is $\pi_1(X)$. If $\pi_1(X)$ is the trivial group, we say S is *simply connected.*

Hence a simply connected space is one for which any closed loop can be continuously contracted to a point.

Exercise 2.4. Show that if $n > 1$, $\pi_n(X)$ is Abelian. Give an example of a topological space X for which the fundamental group $\pi_1(X)$ is non-Abelian.

In fact, the Abelianization of the fundamental group is (for the spaces I shall deal with in this paper) the same as the 1^{st} *homology group* (for whose definition see various of the references).

Definition 2.5. We say two topological spaces X and Y have the same *homotopy type* if there exist maps $f : X \to Y$ and $g : Y \to X$ which are inverses up to homotopy, in the sense that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$; such a map f (and likewise g) is called a *homotopy equivalence*.

Note that for many spaces (*e.g.*, all so-called "CW-complexes", so all spaces I shall need in this paper), a map which induces an isomorphism of all homotopy groups is automatically a homotopy equivalence. Hence one often talks of spaces which are "homotopy n-spheres", that is, spaces which have the same homotopy groups as an n-sphere.

Similarly, there is a weaker notion of a "homology sphere", being a space which has the *homology groups* of a sphere. But the full definition of homology groups is quite complicated (and we don't really need it for the Poincaré Conjecture), so I won't give it in its full glory here. I shall, however, talk a bit about *(de Rham) cohomology* a bit – see §4, below.

Exercise 2.6. Exhibit a homotopy equivalence between $\mathbb{R}^n$ and $\mathbb{R}^m$.

Exercise 2.7. What is the fundamental group of the subset of 3-space formed by the surface of a coffee cup? (Topologists call this a "torus" or "doughnut".) For $j, k = 1, 2, 3$, compute $\pi_j(S^k)$ – work them all out, there are some surprises; some of the generators are famous geometric constructions [*Research:* "the Hopf fibration"]; some of the proofs of rather intuitively obvious parts of this exercise are surprisingly tricky (*e.g.,* when computing $\pi_1(S^2)$, what about the situation mentioned in the above Exercise 1.8?).

Exercise 2.8. *EXTRA CREDIT (and almost certainly a Fields Medal):* For all $j, k \in \mathbb{N}$, compute $\pi_j(S^k)$.

Definition 2.9. A *covering* $p : \tilde{X} \to X$ of a topological space is a surjective continuous map with the property that every point $x \in X$ has a neighborhood U_x such that $p^{-1}(U_x)$ is a disjoint union of sets each of which is mapped homeomorphically onto U_x by p. A *universal cover* of X is a covering whose total space $\tilde{X}$ is simply connected.

Fact 2.10. Most topological spaces X (all those satisfying the mild (and euphoniously named) technical condition of being "semi-locally simply connected", along with being locally path connected and connected) have a universal covering space $\tilde{X}$. The *fibers* $p^{-1}(x)$ for $x \in X$ are all isomorphic to the fundamental group $\pi_1(X)$, and in fact $\pi_1(X)$ acts on $\tilde{X}$ with quotient $X \cong X/\pi_1(X)$.

References 3. Once again, there are many good reference texts on this material: nearly any book with "algebraic topology" in the title is likely to be appropriate. One particularly nice one to try is [May99].

3. Manifolds

Definition 3.1. A *smooth n-manifold* is a topological space X (required for technical reasons to be Hausdorff and second countable, for which definitions see one of the references on topology), an a open cover $\{U_\alpha\}_{\alpha \in A}$ of X, and maps $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ which are homeomorphisms onto their images. The maps are required to satisfy a compatibility condition as follows: for all $\alpha, \beta \in A$, let $U_{\alpha\beta} = U_\alpha \cap U_\beta$; consider

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta}) \qquad ;$$

this map must be have continuous partial derivatives of all orders. The open sets U_α are called *coordinate patches* for the manifold and the maps φ_α are *charts.*

The notion of a manifold is designed to incorporate the concept of a space which is *locally* Euclidean – like enough to Euclidean space that, for example, we can do calculus – but *globally* it may have much more structure, indeed it may have non-trivial topology. The next few definitions are typical of this process of using the locally Euclidean structure for analytic purposes.

Definition 3.2. On a manifold X with coordinate patches U_α and charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ we define the *k-times continuously differentiable functions* $C^k(X)$ to be those functions $f : X \to \mathbb{R}$ which, for each α, give functions $f \circ \varphi_\alpha^{-1}$ mapping the open set $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ to $\mathbb{R}$ that have continuous partial derivatives up to order k. The *smooth functions on* X are $C^\infty(X) = \cap_{k=1}^\infty C^k(X)$.

Definition 3.3. Given two manifolds X and Y, an $f : X \to Y$ is a *smooth map* if for all coordinate patches on Y, f composed with the corresponding chart, when defined, is a vector of $\dim(Y)$ smooth functions. f is a *diffeomorphism* if it is smooth and has a smooth inverse.

One can play this game in other categories: by relaxing the restriction on the composite maps to be merely continuous, one gets a *topological manifold*; requiring the composites to be piecewise linear, one gets a *PL manifold*; requiring real analytic composites, one gets a *real analytics manifold*; *etc.* Today I shall stick to the smooth category.

Definition 3.4. The *tangent space* T_xX at a point x in a manifold X consists of equivalence classes of curves passing through x, where two curves are considered equivalent if they agree to first order in any coordinate patch containing x.

These tangent spaces have the structure of a vector space coming from that on $\mathbb{R}^n$. [As follows: translate the patch so that x corresponds to $0 \in \mathbb{R}^n$, then simply use the scalar multiplication and vector addition of $\mathbb{R}^n$ to get new curves from old; this passes to equivalence classes of curves.]

Exercise 3.5. Exhibit a *canonical* isomorphism $T_x\mathbb{R}^n \cong \mathbb{R}^n$ for any $x \in \mathbb{R}^n$.

Putting together these tangent spaces at all points of X gives

Definition 3.6. The *tangent bundle* TX of a manifold is the union $\bigcup_{x\in X} T_xX$ of all the tangent spaces at the points of X; the projection $p : TX \to X$ takes a vector $v \in T_xX$ to x. ($p : TX \to X$ is actually a vector bundle over X.) A *vector field on* X is a *section* of the tangent bundle, that is, it is a (usually smooth) function $A : X \to TX$ with the property that $p(A(x)) = x$; *i.e.,* for each $x \in X$, $A(x)$ is a vector in the tangent space T_xX.

Exercise 3.7. *[The famous "combing a hairy coconut" problem]* Can you find a smooth, non-vanishing vector field on the 2-sphere? Construct one if you can, prove it is impossible otherwise.

Vector fields allow us to do (a little bit of) calculus on manifolds:

Definition 3.8. If A is a vector field and f a function on a manifold X, then $A \cdot f$ is the function defined at $x \in X$ by $(A \cdot f)(x) = \left.\dfrac{\partial(f\circ\gamma)}{\partial t}\right|_{t=0}$ if $\gamma(t)$ is a curve in the equivalence class defining $A(x)$ for which $\gamma(0) = x$.

References 4. A nice basic book is [War83]. A book which emphasizes the *calculus* aspect is the appropriately named [Spi65]. Milnor's books are masterpieces of exposition: [Mil65] is a basic one which centers on the topological aspects; [Mil69] is about Morse theory, which is beautiful and useful and a good area to explore for students of differential geometry. [Hel01] is a start on the vast theory of Lie groups and related topics.

Definition 3.9. A *Lie group* G is a manifold which is also a group and for which the maps $G\times G \to G : (g, h) \mapsto g\cdot h$ and $G \to G : g \mapsto g^{-1}$ are smooth. When a Lie group G acts on a manifold X, I shall always assume that the map $G \times X \to X$ is smooth.

4. De Rham Cohomology

**This section is optional (and assumes more linear algebra background than the rest).*

Definition 4.1. We need additional notation:

$$\begin{aligned}
\text{for a vector space } V,\quad & \Lambda^p(V) = \{p-\text{multilinear, skew-symmetric maps } V \to \mathbb{R}\}\\
[\text{Note:}\quad & \Lambda^0(V) = \mathbb{R}\ \ \text{(by convention)}\\
\text{and}\quad & \Lambda^1(V) = V^*\ \ \text{(the dual space)}]\\
\text{for } \alpha \in \Lambda^p(V) \text{ and } \beta \in \Lambda^q(V),\quad & \alpha\wedge\beta \in \Lambda^{p+q}(V) \text{ is defined by } (\alpha\wedge\beta)(v_1,\dots,v_{p+q}) =\\
& = \sum_{\sigma\in S_{p+q}} \operatorname{sgn}(\sigma)\alpha(v_{\sigma(1)},\dots,v_{\sigma(p)})\cdot\beta(v_{\sigma(p+1)},\dots,v_{\sigma(p+q)})
\end{aligned}$$

Now,

Definition 4.2. The *bundle* $\Lambda^p(X)$ *of p-forms on a manifold* X is the vector bundle built out of the tangent bundle TX by replacing each fiber T_xX by $\Lambda^p(T_xX)$. The (vector) space of sections of this bundle is denoted $\Omega^p(X)$ and called the space of *p-forms on* X.

Definition 4.3. On a smooth manifold X, there is a (first order differential) operator $d : \Omega^p(X) \to \Omega^{p+1}(X)$ defined by $df(A) = A \cdot f$ for a function $f \in \Omega^0(X)$ and vector field A, and $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \beta \wedge (d\alpha)$ for $\alpha \in \Omega^p(X)$ and $\beta \in \Omega^q(X)$, called the *exterior derivative.*

More importantly, by the linear algebra of skew forms and the equality of mixed partials, $d^2 = 0$.

Exercise 4.4. Prove that $d^2 = 0$.

Hence it makes sense to define

Definition 4.5. The *de Rham cohomology in dimension p of a manifold X* is the vector space

$$H^p_{DR}(X) = \ker(d|_{\Omega^p(X)}) / \operatorname{Im}(d|_{\Omega^{p-1}(X)}) \qquad .$$

The corresponding dimension $b_p = \dim(H^p_{DR}(X))$ is called the *pth Betti number of* X, and the *Euler characteristic* of an n-manifold X is then defined as

$$\chi(X) = \sum_{p=0}^{n} (-1)^p b_p \qquad .$$

Once we have a Riemannian metric on a compact, connected smooth manifold X (see §5, below), there is defined an adjoint operator d^* and a corresponding "Laplacian $\Delta = dd^* + d^*d$ on p-forms", the kernel of which is the space $\mathfrak{H}^p(X)$ of *harmonic p-forms.* Then "Hodge Theory" tells us that $\mathfrak{H}^p(X) \cong H^p_{DR}(X)$.

Exercise 4.6. Prove, starting from the above definition, that the 0^{th} de Rham cohomology of a manifold X obeys $H^0_{DR}(X) = \mathbb{R}^k$, where k is the number of connected components in X.

Exercise 4.7. *EXTRA CREDIT + Research:* Read about "the Mayer-Vietoris exact sequence" (*e.g.,* in [BT82]) and then go compute the de Rham cohomology of all compact 2-manifolds, triangulate these surfaces and compute the traditional Euler characteristic, showing it equals the de Rham version defined above.

References 5. Bott's book [BT82] is quite beautiful. [War83] has good coverage of Hodge theory.

5. Riemannian Geometry

On an n-manifold X we have the "distanceless notion of nearness" inherent in its underlying topology, and some simple calculus from the local Euclidean structure. To go on we want actual distances, which we build up from a smoothly varying inner product on the tangent spaces of X. From this will follow a (distance) metric, and a more refined differentiation operator (the Levi-Civita connection), which in turn yields (one version of) a notion of "straight lines", parallel transport and even *curvature.*

Definition 5.1. A smooth choice of inner products on the tangent spaces of a manifold X is called a *Riemannian metric*; so given vector fields A and B, we write $g(A, B)$ for the function which at $x \in X$ is the inner product of $A(x)$ with $B(x)$ (and the smoothness of g simply means that this function is smooth for smooth vector fields A and B). A manifold with Riemannian metric is called a *Riemannian manifold.*

Definition 5.2. Given a (piecewise C^1) curve $\alpha : [0, 1] \to X$ in a Riemannian manifold X, we define the *length* of α to be

$$\mathcal{L}(\alpha) = \int_0^1 \left(g|_{\alpha(t)} \left(\frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial t} \right) \right)^{\frac{1}{2}} \, dt \qquad .$$

For points x_1 and x_2 in a Riemannian manifold, we define the *distance between* x_1 *and* x_2 to be

$$d(x_1, x_2) = \inf_{\alpha : x_1 \rightsquigarrow x_2} \mathcal{L}(\alpha)$$

where the infimum is over all (piecewise C^1) paths α with $\alpha(0) = x_1$ and $\alpha(1) = x_2$.

So, apparently, an inner product in the fibers of the tangent bundle gives a metric (in the sense of distance function) on the manifold itself.

Definition 5.3. On $\mathbb{R}^n$, define the *usual* or *flat Riemannian metric* by letting the inner product of two vectors in $T_p\mathbb{R}^n \cong \mathbb{R}^n$ be their usual, Euclidean inner product, independently of the point p.

Exercise 5.4. Show that the usual metric on $\mathbb{R}^n$ induces the same distance metric that we usually use.

This gives rise then to generalizations of "straight lines", to wit

Definition 5.5. A curve $\alpha : [01] \to X$ in a Riemannian manifold is called a *geodesic* if for all sufficiently close pairs $t_1, t_2 \in [0,1]$ we have $d(\alpha(t_1), \alpha(t_2)) = \mathcal{L}(\alpha|_{[t_1,_2]})$.

That is, a geodesic is a locally length-minimizing curve.
Here are two more model spaces we shall use again and again:

Definition 5.6. Define the *usual* or *round Riemannian metric* on S^n, viewed as the unit sphere in $\mathbb{R}^{n+1}$, by considering two vectors in $T_pS^n \subset T_p\mathbb{R}^{n+1} = \mathbb{R}^{n+1}$ and taking the (usual) Euclidean inner product in this last vector space.

Definition 5.7. [The upper half-space model of] *n-dimension real hyperbolic space* is, as a manifold, simply the subset $\mathcal{H}^n_{\mathbb{R}} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ of n-dimensional Euclidean space. At a typical point $(x_1, \dots, x_n)$ we give $\mathcal{H}^n_{\mathbb{R}}$ the Riemannian metric which is the (flat) Euclidean metric (inner product) scaled by $\frac{1}{x_n^2}$.

Exercise 5.8. *EASY:* What are the geodesics on $\mathbb{R}^n$ with the flat metric?
HARDER: What are the geodesics for the round metric on the sphere?
HARDEST: What are the geodesics in $\mathcal{H}^n_{\mathbb{R}}$?
[*Research:* "the calculus of variations".]

Another definition of geodesics comes instead from a special notion of differentiation which any manifold can be given:

Definition 5.9. A *connection on* TX is a (differential) operator which takes two vector fields A and B and returns another vector field denoted $\nabla_A B$ which is linear over $\mathrm{C}^\infty(X)$ in the A and over $\mathbb{R}$ in the B, and satisfies $\nabla_A(fB) = (A \cdot f)B + f\nabla_A B$. The *torsion* of such a connection is defined to be $T_\nabla(A, B) = \nabla_A B - \nabla_B A - [A, B]$, where, as usual, $[A, B] = AB - BA$ for any operators A and B.

There is a particularly nice connection in the situation we are most interested in:

Definition 5.10. A Riemannian manifold (X, g) has a unique connection, called the *Levi-Civita connection* with vanishing torsion and for which $\nabla g = 0$ (here we must extend ∇ in a straightforward way to operate on objects like g; equivalently, $A \cdot g(B, C) = g(\nabla_A B, C) + g(B, \nabla_A C)$).

I shall use the Levi-Civita connection without further ado whenever we need a connection on a Riemannian manifold. For example,

Fact 5.11. A curve α in a Riemannian manifold X is a geodesic if and only if $\nabla_{\dot\alpha(t)}\dot\alpha(t) = 0$ [we say the tangent vector is *parallel* along α]. At any x on a complete Riemannian manifold X, there is a map which takes $v \in T_xX$ to $\alpha_v(1) \in X$, where α is a geodesic with $\dot\alpha(0) = v$; this map is called the *exponential map* and is a local diffeomorphism.

Definition 5.12. Given a Riemannian manifold (X, g), a *conformal change of metric* is a metric $e^f g$ on X, where f is some smooth function on X. [This odd exponential formula is used merely to insure that the multiplicative factor between the two metrics is everywhere positive.] Two metrics which are related by such a change are called *conformal*, and a *conformal structure* is a conformal class of metrics.

Exercise 5.13. A punctured 2-sphere can be identified with the plane $\mathbb{R}^2$ by *stereographic projection* ρ*:* set a sphere down on the plane so that its south pole is the point of contact; connect the north pole N to any other point p on the sphere, and continue that line until it hits the plane at some point q; define $\rho(p) = q$. Show that the metric on $\mathbb{R}^2$ coming from its identification via ρ with $S^2 \setminus \{N\}$, endowed with its usual spherical metric, *is not* the usual metric, but *is* conformal to it.

Definition 5.14. A diffeomorphism of Riemannian manifolds which preserves the respective metrics is called an *isometry*. The *isometry group* of a Riemannian manifold is the group of self-diffeomorphisms which are also isometries.

An important invariant of a Riemannian manifold is

Definition 5.15. The *curvature tensor of a Riemannian manifold* X is (the endomorphism-valued 2-form) defined as the vector field $R(A, B)C = \nabla_{[A,B]}C - [\nabla_A, \nabla_B]C$ for vector fields A, B and C on X. The *sectional curvature* of a 2-plane $\sigma \subseteq T_xX$ with basis $\{A, B\}$ is $K(\sigma) = g(R(A, B)A, B)/(g(A, A)g(B, B) - g(A, B)^2)$. The *Ricci curvature* of X is the 2-tensor *Ric* which on vector fields A and B gives the function $r(A, B) = \mathrm{tr}(C \to R(A, C)B)$. The *scalar curvature* is the function s on X defined by $s = \mathrm{tr}_g(r)$.

Exercise 5.16. Compute, in as explicit as possible a manner, all of the objects/maps/operators/invariants/groups defined in Definition 5.10, Fact 5.11, and Definitions 5.14 & 5.15, for the Riemannian manifolds: $\mathbb{R}^n$, S^n, and $\mathcal{H}^n_{\mathbb{R}}$. (Warm up with $n = 2$, first.)

A concept we shall need below is that of

Definition 5.17. A *Riemannian homogeneous space* is a Riemannian manifold whose isometry group acts transitively.

References 6. Standard, fine introductions to Riemannian geometry are [dC92], [Lan85]. More exhaustive[/ing] books are the classic [Spi99] and the quite new [Ber03]. The [Hel01] already mentioned above also covers a lot of this material, both generally and also in particular with emphasis on homogeneous spaces and their geometric properties.

6. Dimension Two

The situation in two dimensions is complex (pardon the pun) and beautiful, with many classical results and yet amazing current research. It should be studied and understood by all students of mathematics!

In this section, Σ will be a compact, connected manifold of dimension two.

Definition 6.1. Σ is said to be *orientable* if is possible to choose a global non-vanishing 2-form on Σ. This amounts to being able to make a consistent choice of the notion of "counterclockwise rotation" in each of the tangent spaces of Σ.

A conformal class of Riemannian metric on an orientable Σ allows us to identify each tangent space, which can already be (non-canonically) identified with $\mathbb{R}^2$, in fact with $\mathbb{C}$: an orientation together conformal structure lets us construct an operator in each tangent space which is "rotation by angle $\pi/2$ in a counterclockwise direction – which is exactly what we need to be the operation of multiplication by i. This endows Σ with a *complex structure.*

References 7. A nice place to start in studying complex manifolds is the (oddly, modestly named) [Che79].

We shall now assume our Σ is in fact oriented, and we shall refer to it as a *Riemann surface*, which terminology historically of often associated with the complex structure whose existence I just mentioned.

Definition 6.2. On our 2-dimensional Σ, the sectional curvature, defined above, has only one value at each point $x \in \Sigma$ [T_xX is *itself* a 2-plane], hence defines a function which is called the *Gaussian curvature* and written K as well.

The following is a very classical theorem (known to Gauss himself!), which can be proven in a number of ways.

The Gauss–Bonnet Theorem 6.3. For an arbitrary metric on our Σ,

$$\int_\Sigma K \; d\mathrm{vol} = 2\pi\chi(\Sigma)$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ defined as above in §4 or in classical terms as $V-E+F$ for a triangulation.

Definition 6.4. Instead of the Euler characteristic of a Riemann surface Σ, one often speaks of its *genus*, which can be defined either as $g = \frac{2-\chi(\Sigma)}{2}$ or in an entirely different way using the complex structure. [With either definition, the genus intuitively represents the number of holes in the "doughnut".]

A nice application of non-linear global analysis gives:

Fact 6.5. Any metric on our Σ can be conformally changed to have constant scalar (Gaussian) curvature of -1, 0 or 1, depending upon the sign constraint imposed by the Gauss-Bonnet Theorem and $\chi(\Sigma)$.

This is then the geometric analyst's approach to what the complex analyst would call "Uniformization": we pick a random Riemannian metric on Σ. We can then compute the genus of Σ from the Gauss-Bonnet Theorem. In particular,

- If Σ has genus 0 – so it is topologically the sphere, so *simply connected* – the metric can be corrected to be the round metric on S^2.
- If Σ has genus 1, the metric can be made flat, so the universal cover of Σ is isometric to $\mathbb{R}^2$ with its usual metric. Topologically, Σ is the torus, so $\pi_1(\Sigma) \cong \mathbb{Z}^2$ and $\Sigma \cong \mathbb{R}^2/\mathbb{Z}^2$.
- If the genus is 2 or greater, the metric can be made constant curvature, equal to -1. Thus $\Sigma \cong \mathcal{H}^2_{\mathbb{R}}/\pi_1(\Sigma)$.

In each case, the key is to correct the metric to one of constant curvature, whereupon since we know the simply-connected, constant-curvature spaces well (they are S^2, $\mathbb{R}^2$ and $\mathcal{H}^2_{\mathbb{R}}$ in dimension 2), we can describe our particular Σ as the isometric and topological quotient of the appropriate model space by the action of the fundamental group.

Exercise 6.6. Think of the points of $\mathcal{H}^2_{\mathbb{R}}$ as complex numbers, so $\mathcal{H}^2_{\mathbb{R}} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Show that the group $PSL(2,\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) = 1\}/\{\pm Id_{2\times 2}\}$ acts *isometrically* on $\mathcal{H}^2_{\mathbb{R}}$ by fractional linear transformations $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$. Then describe geometrically how particular isometries in this group act on $\mathcal{H}^2_{\mathbb{R}}$.

Exercise 6.7. Let Σ be a Riemann surface of genus g – so, "the surface of a doughnut (maybe pretzel?) with g holes". Show that $\pi_1(\Sigma)$ is isomorphic to

$$F_{A_1,\dots,A_g,B_1,\dots,B_g} / \left\langle \Pi_{j=1}^g [A_j, B_j] = id \right\rangle$$

i.e., the free group on $2g$ generators $A_1, \dots, A_g, B_1, \dots, B_g$ modulo the subgroup generated by the one relation $\Pi_{j=1}^g [A_j, B_j] = id$. [*Research:* "the Van Kampen Theorem".]

Exercise 6.8. Find (many examples of) a polygon in $\mathcal{H}^2_{\mathbb{R}}$ with $4g$ sides, for $g \in \mathbb{N}$, along with $2g$ isometries in $PSL(2,\mathbb{R})$ which identify these sides in pairs... so that a Σ of genus g with its constant negative curvature metric is the quotient of $\mathcal{H}^2_{\mathbb{R}}$ by the action of the group generated by your $2g$ isometries.

References 8. A wonderful book on material related to Exercise 6.8 is [Kat92]. Another good one is [Jos06]. While it covers much beyond two dimensions, it is interesting to note that [Spi99], mentioned above, is known to (some) students as "all the way with Gauss-Bonnet". For Fact 6.5, see [Kaz85]. The complex function theory version of some of the results of this section was done by Koebe in [Koe07] and Poincaré himself in [Poi07].

In summary, then, the method I have sketched here is via *special metrics*: one chooses any random metric on Σ and then does the (non-linear, global) analysis to reform this to have some particularly nice property (here, it becomes constant curvature). Then since the simply connected examples with this property are well understood (here, they are S^2, $\mathbb{R}^2$ or $\mathcal{H}^2_{\mathbb{R}}$), one has has Σ as the quotient of one of these model spaces by $\pi_1(\Sigma)$ and Σ is thus very well understood topologically – and even has a nice model geometry.

This is by no means the only point of view on for two-manifolds: there is, for example, the very beautiful approach based on complex function theory. Nevertheless, it seems so promising that it begs for generalization to higher dimensions. Oddly enough, the most difficult dimensions are the next few after two, while in much higher dimensions there are strangely powerful techniques. I shall next explain the special issues in three dimensions and the advances which have so recently found traction. Then in a final §8, I shall briefly mention the results for dimensions greater than three.

7. Dimension Three

As a first step, we would like some potential list of "all" three-manifolds, perhaps according to a coarse, topological invariant (analogous to the Euler characteristic in dimension two). However, a brief examination of some examples shows that 3-manifolds can in general be quite a bit too complicated for a simple classification: for example, they often can be induced to come apart into pieces, each of which potentially has its own complexity (and, ultimately, geometry). We examine these decompositions first.

Definition 7.1. A *connected sum* of two 3-manifolds X_1 and X_2 is a third 3-manifold formed by puncturing X_1 and X_2 and then identifying small neighborhoods of the punctures (each of which is essentially a punctured ball in $\mathbb{R}^3$) in the separate pieces; this identification is done in such a way that passing into what used to be a sphere around the puncture in X_1 amounts to passing out of the sphere which used to go around the puncture in X_2. A manifold is *prime* if there is no non-trivial way to write it as a connected sum. [A *trivial connected sum* is a connected sum of some manifold with S^3.]

[There is a very closely related notion to primality called *irreducibility* which is used in some versions of the results in the section; for a comparison, see any basic reference on the topology of 3-manifolds – for example, [Hat00] is a nice on-line resource.]

Definition 7.2. A torus T embedded in a 3-manifold X is called *incompressible* if T is orientable and the inclusion map induces an injection on fundamental groups π_1; *i.e.,* if every loop which is homotopically nontrivial in T remains nontrivial even up to homotopy in the ambient X.

Fact 7.3. Every compact, connected 3-manifold admits a (finite!) maximal connected sum decomposition into prime pieces. [This is the *Sphere (or Prime) Decomposition*, due to Kneser and Milnor.] Every compact, connected prime 3-manifold admits a finite maximal collection of disjoint incompressible tori. [This is *Torus Decomposition* of Jaco-Shalen.]

The Thurston Geometrization Conjecture 7.4. Decompose at compact, connected 3-manifold by the Sphere and Torus Decompositions. Then the resulting fragments each can be endowed with of one of the following eight types of geometries:

(1) Euclidean (flat) geometry, whose isometry group is the group of rigid motions of $\mathbb{R}^n$;
(2) 3-dimensional hyperbolic geometry (with constant negative curvature) having isometry group $PSL(2,\mathbb{C})$;
(3) spherical geometry (with constant positive curvature), isometry group $O(3)$;
(4) the geometry of $S^2 \times \mathbb{R}$;
(5) the geometry of $\mathcal{H}^2 \times \mathbb{R}$;
(6) the geometry of the universal cover of the group $SL(2,\mathbb{R})$;
(7) Nil geometry – the geometry of the Heisenberg group (the group of matrices of the form $\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$); or
(8) Sol geometry (the geometry of the semidirect product of $\mathbb{R}$ with $\mathbb{R}^2$, where the former operates on the latter by $(z,(x,y)) \mapsto (e^z x, e^z y)$).

[The isometry groups in the last three of these cases is simply the named group, while for cases (4) and (5) it is a simple modification of the corresponding isometry group in two dimensions. Also, note that the torus decomposition of the Conjecture is slightly different from Jaco-Shalen's, in certain easily identifiable cases.]

References 9. The book [TLe97] by William Thurston himself (with help from Silvio Levy) puts a lot of this together, also with much background in 2-manifolds and connections to other areas.

A consequence of the Geometrization Conjecture, through an analysis of which decomposition pieces must occur and the properties of the spherical case (3), yields

The Poincaré Conjecture 7.5. A compact, connected, simply-connected 3-manifold is necessarily the 3-sphere.

Let us clarify a point in the statement of the Geometrization Conjecture:

Definition 7.6. To endow a three-manifold X with a geometric structure based on some group G means that a diffeomorphism is given of X with a Riemannian homogeneous space $\Gamma\backslash G/K$, where K is a compact subgroup of the isometry G and Γ is a discrete group acting freely on G/K.

Perelman's Proof 7.7. The Poincaré Conjecture – and, in fact, also the Geometrization Conjecture – was finally proved by Grisha Perelman in e-prints from 2002 and 2003 which fulfilled the promise of an approach suggested by Richard Hamilton two decades earlier. Hamilton's idea was much like the "special metric" method described in the previous section: he imagined starting with an arbitrary Riemannian metric on the 3-manifold X and reforming it by solving a non-linear PDE. The particular PDE he used was the one which described the *Ricci flow*, that is, he considered the PDE

$$\frac{dg}{dt} = -2Ric_g$$

for the metric g itself. Analytically, this is a parabolic equation (something like a heat equation), with many attractive features (*e.g.,* heat equations are smoothing, as even a very rough distribution of heat quickly spreads out to be smooth).

Hamilton's original work on the Ricci flow showed there are solutions at least for short times. In fact, Hamilton was able to show that if the Ricci curvature of the metric at time $t = 0$ were everywhere positive, there would be solutions for all time, and the solution would converge to a metric of constant curvature. However, if X had some negative curvature – the specific concrete example was of two round spheres S^3 connected by a thin neck – then the Ricci flow could blow up in finite time; another particularly bad solution was called the "cigar soliton".

Perelman gets around these finite-time singularities in the Ricci flow by proving that they are only of certain standard types. Then a surgery can be performed around the singularity and the Ricci flow restarted. Now as the time goes to infinity, the metric splits X (along incompressible tori!) into a *thick* part with, in the limit, hyperbolic metric of finite volume, and a *thin* part which converges to a *graph manifold*, for which the Thurston geometrization is understood.

Exercise 7.8. *Research:* Read Hamilton's paper [Ham88] to learn about the Ricci flow, in the context of using it to prove the uniformization of Riemann surfaces.

Here is a project on a topic which is of course enormously different from the Ricci flow on 3-manifolds, but which has something of the same spirit of "reforming towards a special geometry":

Exercise 7.9. *Research/project:* Read about the "curve-shortening flow". Write a piece of software which allows a user to sketch a curve and which then traces the evolution under this flow.

References 10. Perelman announced his work in the three e-prints [Per02], [Per03b] and [Per03a]. After a little while a number of summaries and explications of his work appeared, including the very detailed [KL06], [MT06] (which is now also in book form as [MT07]) and [CZ06], and many higher-level surveys such as [Mil03] and [And04]. A host of elaborations, applications and (modest) simplifications (in special cases) of Perelman's work continues to appear in journals and on the 'net (search for "Perelman" on `http://www.arXiv.org`). Hamilton's papers [Ham82], [Ham95], and [Ham99] are the foundational works on his approach to the Ricci flow. There is also a nice explanation of some intuition for the Ricci curvature in [Bes87]. For the terms "thick", "thin" and "graph manifold", see the [TLe97] already referenced. For the curve-shortening flow, see [CZ01].

8. Dimensions Four and Greater

What about higher dimensions? Oddly enough, the situation is a bit easier in certain respects because there is so much *room* in higher dimensional spaces. For example, consider a relation in the fundamental group of a manifold X of dimension five or greater. Such a relation amounts to a homotopy of a loop to the trivial loop, so is some kind of disk sitting inside X. But with all those dimensions to work with, pairs of 2-dimensional disks in X can be moved so that they are disjoint and have no self-intersections. A whole variety of steps with this general flavor are possible in higher dimensions.

Next, note that there are simple examples which show that the hypothesis of a new conjecture must not be based merely on the fundamental group. Note however that a compact, simply connected 3-manifold in fact is a homotopy 3-sphere, and this is the hypothesis it makes sense to generalize:

The Generalized Poincaré Conjecture 8.1. A compact manifold of dimension n with the homotopy type of the sphere S^n *is* S^n

Of course, this statement begs a question (familiar to followers of American presidential politics in the 1990's): what is the meaning of "is"? There is really an issue here, since in three dimensions, the theories of smooth and topological (and even "piecewise linear", called "PL" by topologists) manifolds are identical – but not so in higher dimensions. Therefore it should not be surprising that there are different results, which have technically different hypotheses and conclusions, in dimensions ≥ 4.

Historically the first result in this direction was due to Stephen Smale in 1961:

Theorem 8.2. *If X is a smooth homotopy sphere of dimension $n \geq 5$ then X is homeomorphic to S^2.*

Despite the conclusion being the existence of a homeomorphism, Smale used smooth techniques, very much like those around the "h-cobordism theorem". [Wallace proved a similar result, at least for $n \geq 6$, shortly after Smale.]

Shortly thereafter, Stallings used very different methods and, helped by Zeeman particularly in dimensions 6 and 7, proved

Theorem 8.3. *If X is a compact PL manifold of dimension $n \geq 5$ with the homotopy type of S^n then X is homeomorphic to S^n. (In fact, the homeomorphism is PL except possibly at one point.)*

Dimension four was more difficult. Progress came in the early 1980s from Michael Freedman, who gave a complete topological classification of simply connected, topological 4-manifolds by two algebraic topological invariants, the intersection form and the "Kirby-Siebenmann invariant" (based on very non-smooth constructions). The conclusion then was

Theorem 8.4. *If X is a compact topological 4-manifold with the homotopy type of S^4 then X is homeomorphic to S^4.*

The smooth version in dimension four has so far resisted all approaches. In fact, the smooth theory of 4-manifold is very surprising. For example, work first done by Simon Donaldson in the mid-1980s shows that there are manifolds which are homeomorphic to $\mathbb{R}^4$ but *not diffeomorphic* – in fact, there are uncountably many non-diffeomorphic ones. Apparently, in dimension four there is very rich structure here which has in many ways remained a mystery since the mid-80s, despite a new, far simpler approach found by Nathan Seiberg and Edward Witten to much of the theory based on Donaldson's work.

After the 60s and then 80s showed such progress on the Generalized Poincaré Conjecture, the original Conjecture remained open until the innovations of Grisha Perelman from around the turn of the millennium. But now low-dimensional topology and geometry (at least in dimension three: four is still a problem) stands ready for a revolution based on harvesting the fruits of the Poincaré and Geometrization Conjectures and of the intriguing intricacies of Perelman's proof. It will be fascinating to see what happens in the next few years.

References 11. There are several surveys of the issues around the Generalized Poincaré Conjecture such as [Mil] (and references therein). Smale's version appeared in [Sma61], Wallace's work in [Wal60] and [Wal61], Stallings' in [Sta60], Zeeman's in [Zee60] and [Zee61], and Freedman's in [Fre82]. Probably the best places to read about Donaldson's work are [DK90] and [FU91], while the best for the Seiberg-Witten version is [Mor95].

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Department of Mathematics and Physics, Colorado State University, Pueblo, 2200 Bonforte Blvd., Pueblo, CO 81001
E-mail address: `jonathan.poritz@gmail.com`
`www.poritz.net/jonathan`